Extremal eigenvalues of critical Erdos Renyi graphs

Raphael Ducatez (joint work with Johannes Alt, Antti Knowles)

$\textit{Mach}, 2022. \text{CY Days in } \underbrace{\text{Nonlinear Analysis}}_{\text{"Matrices \& Probability"}},$

References

1. Extremal eigenvalues of critical Erdős-Rényi graphs,(arXiv:1905.03243)
2. Delocalization transition for critical Erdős-Rényi graphs (arXiv:2005.14180)
3. Poisson statistics and localization at the spectral edge of sparse Erdős-Rényi graphs (arXiv:2109.03227).
4. The completely delocalized region of the Erdős-Rényi graph


(joint work with Johannes Alt, Antti Knowles)

Outline

1. Model and results
2. Existence of the extremal eigenvalues
3. Delocalization transition
4. Strong localization and fluctation of the largest eigenvalue.

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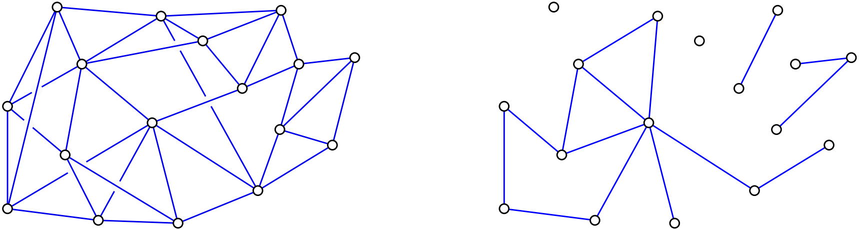
The E-R model.

$A = (A_{xy})_{x,y \in [N]} \in \{0,1\}^{N \times N}$ is the adjacency matrix of the homogeneous Erdős-Rényi graph

- N vertices
- each edge $e \in G$ with probability $p_N = \frac{d_N}{N}$.



We also consider the "centered matrix" $\underline{A} = A - \mathbb{E}(A)$.

Question : what can we say about its eigenvalues?

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- N vertices
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$$A = \begin{pmatrix} 0 & 1 & 0 & & & 1 \\ 1 & 0 & & & & \\ 0 & & & & & \\ & & & & \ddots & \\ & & & & & \\ 1 & & & & & \end{pmatrix}$$

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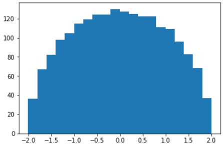
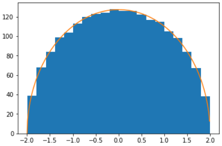
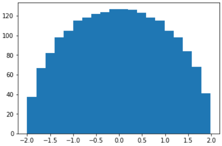
Dans le cas symmétrique, qu'est ce qu'on peut dire du spectre de X?
(Ou plutôt de $\frac{1}{\sqrt{N}}X$ car $\mathbb{E}(\frac{1}{N}\mathrm{Tr}(X_{ij}^2)) = N$)



(X_{ij} uniform)



(X_{ij} normal law)



(X_{ij} Bernoulli law)

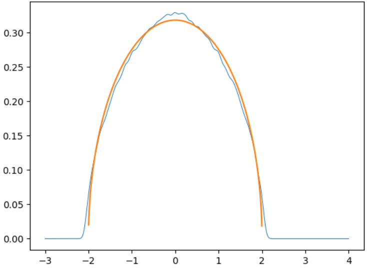
Semi-circle law

The density of the eigenvalues of $\frac{1}{\sqrt{N}}X$ converge to the semi-circle law

$$\mu_{sc}(x) = \frac{\sqrt{4-x^2}}{2\pi}1_{[-2,2]}(x)dx$$

The semi-circle law

In the regime $d \equiv d_N \to \infty$ as $N \to \infty$, the empirical eigenvalue measure of $A/\sqrt{d}$ converges to the semicircle law supported on $[-2,2]$.



Question : Are there any eigenvalues outside the bulk $[-2,2]$?

Statistics of the degree of the vertices in the critical regime $d = b\log(N)$

Connectivity transition in the Erdos Renyi graph:

- If $d \leq \log(N) - ...$, there exists some isolated vertices.
- If $d \geq \log(N) + ...$, there is no isolated vertices.

Degree statistic : (Consider N, iid Poisson variables of parameter d)

- For any x and $\alpha > 1$

$$\mathbb{P}(D_x \geq \alpha d) \approx \exp(-dh(\alpha)) = N^{-b \cdot h(\alpha)}$$

  with $h(\alpha) = \alpha \log(\alpha) - \alpha + 1$, $d = b \cdot \log(N)$.

- The number of large degree vertices is

$$\#[x \in [N] : D_x \geq \alpha d] \approx N^{(1 - b \cdot h(\alpha))}$$

- The maximal degree $D_{\max} = \alpha_{\max} d$ satisfies $1 - b \cdot h(\alpha_{\max}) = 0$.

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Statistics of the degree of the vertices in the critical regime $d \propto \log(N)$

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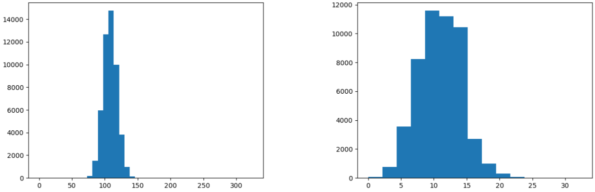

Degree distribution for $d = 10 \log N$ and for $d = \log N$

Main (first) result

- $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 2 + o(1)$ the largest eigenvalue of $\underline{A}/\sqrt{d}$,
- $\lambda_N \leq \cdots \leq \lambda_{N-l+1} \leq -2 - o(1)$ the smallest eigenvalues.
- $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l \geq 2$ the largest degrees ($D_i = \alpha_i d$) of the E-R graph.

Correspondance large eigenvalue-large degree (D, Alt, Knowles)

For all $i \leq l$

$$|\lambda_i - \Lambda(\alpha_i)| \leq o(1),$$

$$|\lambda_{N-i+1} + \Lambda(\alpha_i)| \leq o(1),$$

with $\Lambda(\alpha) = \frac{\alpha}{\sqrt{\alpha - 1}}$.

Corollaire : Transition for the spectrum

There exists eigenvalues outside of the bulk iif $\alpha_{\max} > 2$ iif $d < d_* = \frac{1}{2\log(2)-1}\log(N) \approx 2.58\cdots\log(N)$.

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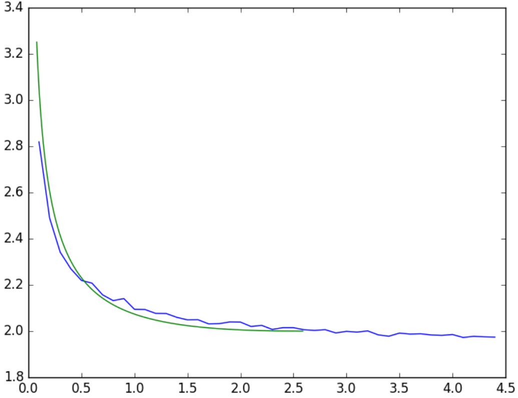
A small numerical simulation.

We can calculate $\Lambda(\alpha_{\max})$ with $\alpha_{\max}$ solution of $1 - ch(\alpha_{\max}) = 0$.

Figure: The largest eigenvalue with $N = 1000$, $d = c\log(N)$ and the theorical prediction.



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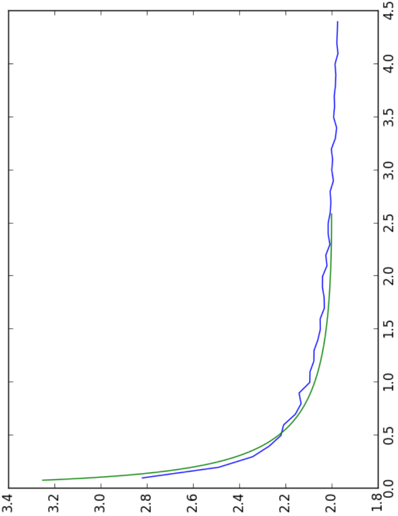
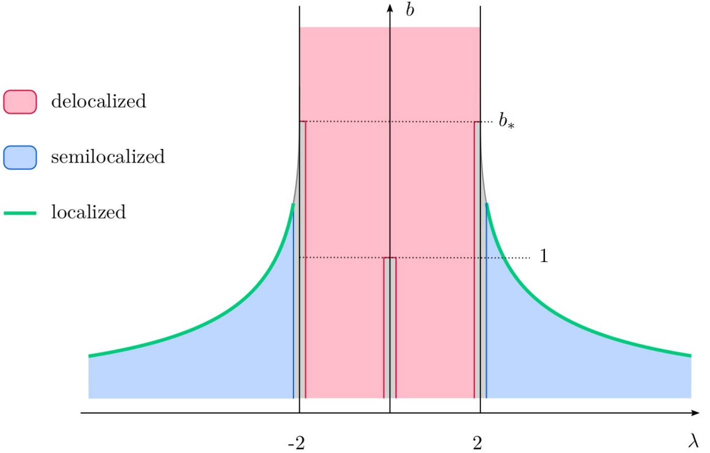
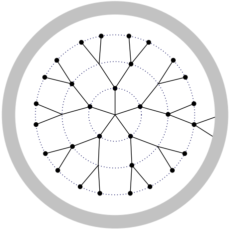


Main results : The localized and delocalized spectrum



Outline



With the basis $(1_{S_i(x)}/\|1_{S_i(x)}\|)_i$:

$$\frac{1}{\sqrt{d}}U^{-1}AU = \begin{pmatrix} 0 & \sqrt{\alpha_x} & & \\ \sqrt{\alpha_x} & 0 & 1 & \\ & 1 & 0 & \ddots \\ & & \ddots & 0 \end{pmatrix}$$

Lemma

Its spectrum is

- $[-2,2]$ if $\alpha_x \leq 2$,
- $\{\pm\Lambda(\alpha_x)\} \cup [-2,2]$ if $\alpha_x > 2$.

with $\Lambda(\alpha) = \frac{\alpha}{\sqrt{\alpha-1}}$. Moreover in the seconde case the corresponding eigenvector (u) satisfies $u_i = \gamma_\lambda^{i-1} u_1 \, , |\gamma_\lambda| < 1$.

(semi-localization) In the E-R graph

Proposition

The Erdos Renyi graph on the ball of radius r is "close" to the regular tree.

Let x with $D_x > 2d$. We define with $S_i(x)$ the sphere in the Erdos Renyi graph,

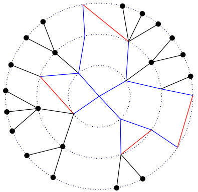
$$u = \sum_{i \le r} u_i \frac{1_{S_i(x)}}{\sqrt{|S_i(x)|}}$$

A candidate eigenvector

$$\|(\frac{1}{\sqrt{d}} A - \Lambda(\alpha_x))u\| = o(1)$$

Corollary

There exists an eigenvalue λ of $\frac{1}{\sqrt{d}} A$ with $|\Lambda(\alpha_x) - \lambda| = o(1)$.



Moment method

For B the nonbacktracking matrix associated with $\underline{A}/\sqrt{d}$

$$\mathbb{E}(\mathrm{Tr}(B^l(B^*)^l)) = O((1+o(1))^l)$$

for all $l \sim \sqrt{d}\log(n)$

Corollary

$$\rho(B) \leq 1 + o(1)$$

An "Ihara-Bass formula" Spectrum of $B \leftrightarrow$ Spectrum of A

Outline

Localization or Delocalization ?

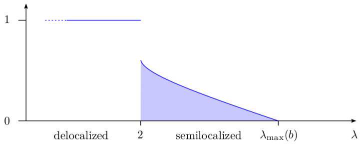
- The Anderson Model.
  - Physical prediction : Anderson (1958), Mott (1960's)
  - Proof of localization at the edge and at strong disorder : J. Fröhlich-T. Spencer (1983), Aizenman-Molchanov (1993).
  - Proof of delocalization still OPEN.
- Random matrices.
  - (Trivial) Delocalization for Gaussian matrices.
  - For generalized Wigner matrix : Erdos-Yau-... (2009-...). Sparce random matrices (Knowles)
- Band matrices
  - localization/Delocalization transition predicted $B \sim \sqrt{N}$
- On Trees
  - localization and delocalization phases M. Aizenman and S. Warzel (2006).

[A.,D.,K.] Delocalization transition of critical Erdős-Rényi graphs

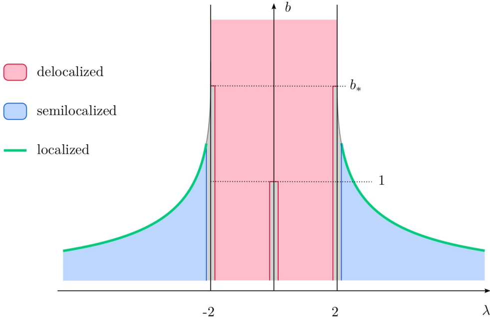
Phase transition

Let u an eigenvector with eigenvalue λ:

- (Delocalized Phase) For λ outside $[-2,2]/\{0\}$ then $\|u\|_{L^\infty}^2 = \mathcal{O}(N^{-1+o(1)})$.
- (Semilocalized Phase) For $|\lambda| > 2$ then $\|u\|_{L^\infty}^2 \geq N^{-\rho(\lambda)+o(1)}$.



$\rho(\lambda)$



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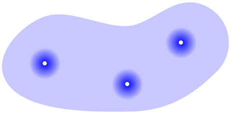
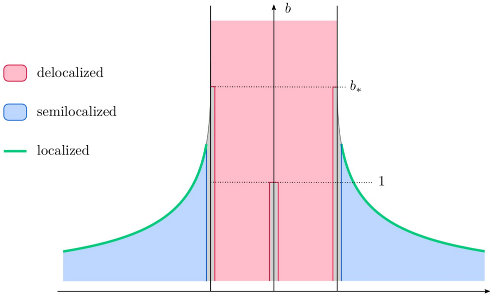
- (Semilocalized Phase) For $|\lambda| > 2$, u is a linear combination of vectors supported on balls around the vertices of large degree.





(delocalization phase) A local law

For $z \in \mathbb{C}, \Im(z) = \eta > 0,\ G = G(z) = (\frac{1}{\sqrt{d}} A - z)^{-1}$

$$\eta^{-1} |\phi_i(x)|^2 \leq \max_{\Re z \in \mathbb{R}} \Im \sum_j \frac{|\phi_j(x)|^2}{(\lambda_j - z)} = \max_{\Re z \in \mathbb{R}} \Im G_{xx}$$

Local law

For all $z \in \mathbb{C}$ with $\Re z \in (-2+\varepsilon, -\varepsilon) \cup (\varepsilon, 2-\varepsilon)$ and $\Im z > N^{-1+\varepsilon}$ we have

$$\max_{x,y} |G_{xy}(z) - \delta_{xy} m_{\alpha_x}(z)| = o(1)$$

where $m(z) = -\frac{1}{z+m(z)}$ and $m_\alpha(z) = -\frac{1}{z+\alpha m(z)}$.

Corollary

$|\phi_i(x)|^2 = O(N^{-1+\varepsilon})$.

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Corollary

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Schur complement formula

$$\frac{1}{G_{xx}} = -z - \frac{1}{d} \sum_{y \in S_1(x)} G_{yy}^{(x)} + o(1), \qquad \frac{1}{m(z)} = -z - m(z).$$

"Local law steps"

1. Here one has to restrict to typical vertices $\mathscr{T} \subset [N]$.
2. $$\left| \frac{1}{d} \sum_{y \in S_1(x)} G_{yy} - \frac{1}{N} \sum_{y \in [N]} G_{yy} \right| = o(1)$$
3. This implicite solution is "stable" around $G_{xx} = m(z)$.

Corollary

$$\left| \frac{1}{N} \sum_{y \in [N]} G_{yy} - m(z) \right| = o(1)$$

Outline

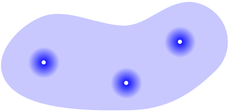
Fluctuation of the largest eigenvalue

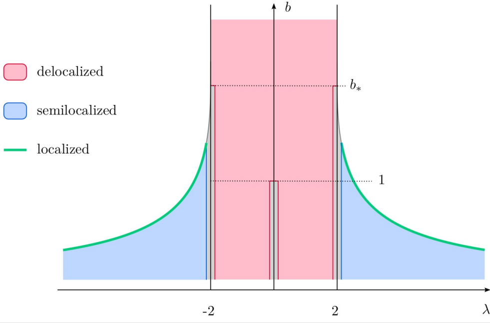
We define $dh(\alpha_{\max}) = \log N$ and $\Lambda_{\max} := \Lambda(\alpha_{\max})$.

Theorem

Around $\Lambda_{\max}$ the spectrum of converge to a Poisson point process. The density is explicit and we have

$$\mathbb{P}(\frac{\lambda_1 - \Lambda_{\max}}{d} \leq t) \to \exp(-F(t))$$





Strong localization?

Localization

(Work in progress) Do we have complete localization in the semilocalized phase ?

Thank you for your attention