Global regularity for 2D Navier-Stokes free boundary with small viscosity contrast

Francisco Gancedo
University of Seville

CY Days in Nonlinear Analysis

Paris, March 30, 2022

- Global Regularity of 2D Navier-Stokes Free Boundary with Small Viscosity Contrast
  with Eduardo García-Juárez (Preprint 2021)

- Quantitative Hölder estimates for even singular integral operators on patches
  with Eduardo García-Juárez (Preprint 2021)

Incompressible fluid interface problems:

- Vortex patch

Incompressible fluid interface problems:

- Vortex patch
- Water waves

Incompressible fluid interface problems:

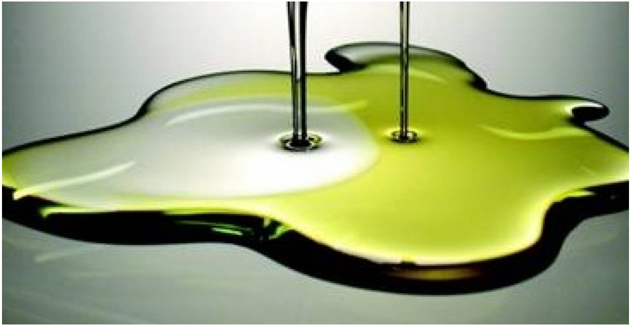
- Vortex patch
- Water waves
- Muskat problem

Incompressible fluid interface problems:

- Vortex patch
- Water waves
- Muskat problem
- Sharp front evolution

Incompressible Navier-Stokes dynamics

$$\nabla \cdot u(x,t) = 0,$$
$$u_t(x,t) + u(x,t) \cdot \nabla u(x,t) = -\nabla P(x,t) + \mu \Delta u(x,t)$$



N-S free boundary (N-SFB)

$$\begin{cases} \nabla \cdot u(x,t) = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0, \\ \rho_t(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0, \\ \\ \\ \\ \end{cases}$$

N-S free boundary (N-SFB)

$$\begin{cases} \nabla \cdot u(x,t) = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0, \\ \rho_t(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0, \\ (\rho u)_t(x,t) + u(x,t) \cdot \nabla(\rho u)(x,t) = -\nabla P(x,t) + \nabla \cdot (\mu \mathbb{D} u)(x,t), \\ \\ \\ \end{cases}$$

N-S free boundary (N-SFB)

$$\begin{cases} \nabla \cdot u(x,t) = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0, \\ \rho_t(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0, \\ (\rho u)_t(x,t) + u(x,t) \cdot \nabla(\rho u)(x,t) = -\nabla P(x,t) + \nabla \cdot (\mu \mathbb{D} u)(x,t), \\ \mu_t(x,t) + u(x,t) \cdot \nabla \mu(x,t) = 0, \\ \mathbb{D}u(x,t) = (\partial_j u_k(x,t) + \partial_k u_j(x,t)), \end{cases}$$

N-S free boundary (N-SFB)

$$\begin{cases} \nabla \cdot u(x,t) = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0, \\ \rho_t(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0, \\ (\rho u)_t(x,t) + u(x,t) \cdot \nabla (\rho u)(x,t) = -\nabla P(x,t) + \nabla \cdot (\mu \mathbb{D} u)(x,t), \\ \mu_t(x,t) + u(x,t) \cdot \nabla \mu(x,t) = 0, \\ \mathbb{D} u(x,t) = (\partial_j u_k(x,t) + \partial_k u_j(x,t)), \end{cases}$$

where density and viscosity are given by

$$(\rho, \mu)(x,t) = \begin{cases} (\rho^{in}(x), \mu^{in}(x)), & x \in D(t) \subset \mathbb{R}^2, \\ (\rho^{out}(x), \mu^{out}(x)), & x \in \mathbb{R}^2 \smallsetminus \overline{D(t)}, \end{cases}$$

and

$$D(t) = X(D(0),t), \quad X_t(a,t) = u(X(a,t),t), \quad X(a,0) = a, \quad a \in \mathbb{R}^2.$$

N-S free boundary (N-SFB)

Weak formulation≡Boundary conditions:

$$u^{in}(x,t) = u^{out}(x,t), \quad x \in \partial D(t),$$

$$(\mu^{in}\mathbb{D}u^{in} - \mathbb{I}P^{in})(x,t)n(x,t) = (\mu^{out}\mathbb{D}u^{out} - \mathbb{I}P^{out})(x,t)n(x,t), \quad x \in \partial D(t).$$

N-SFB: Previous results

One-fluid case ($\mu^{out} = 0 = \rho^{out}$):

- Local existence:
  Solonnikov-77 (closed domain)
  Beale-81 (with gravity force)

N-SFB: Previous results

One-fluid case ($\mu^{out} = 0 = \rho^{out}$):

- Local existence:
  Solonnikov-77 (closed domain)
  Beale-81 (with gravity force)

- Global regularity (small initial velocity and profile):
  Beale-81 (up to a time $T > 0$)
  Sylvester-90
  Tani, Tanaka-95
  Danchin, Hieber, Mucha, Tolksdorf-20 (low regularity)

N-SFB: Previous results

One-fluid case ($\mu^{out} = 0 = \rho^{out}$):

- Local existence:
  Solonnikov-77 (closed domain)
  Beale-81 (with gravity force)

- Global regularity (small initial velocity and profile):
  Beale-81 (up to a time $T > 0$)
  Sylvester-90
  Tani, Tanaka-95
  Danchin, Hieber, Mucha, Tolksdorf-20 (low regularity)

- Sharp decay rates:
  Guo, Tice-13[2]

N-SFB: Previous results

One-fluid case ($\mu^{out} = 0 = \rho^{out}$):

- Local existence:
  Solonnikov-77 (closed domain)
  Beale-81 (with gravity force)

- Global regularity (small initial velocity and profile):
  Beale-81 (up to a time $T > 0$)
  Sylvester-90
  Tani, Tanaka-95
  Danchin, Hieber, Mucha, Tolksdorf-20 (low regularity)

- Sharp decay rates:
  Guo, Tice-13[2]

- Finite time blow-up:
  Castro, Córdoba, Fefferman, **G.**, Gómez-Serrano-15 (2D)
  Coutand, Shkoller-15 (3D)

Two-fluid interaction:

- Local existence:
  Denisova-94,01

Two-fluid interaction:

- Local existence:
  Denisova-94,01

- Global regularity (small initial velocity):
  Denisova-07
  Saito, Shibata, X. Zhang-20 (low regularity)

Two-fluid interaction:

- Local existence:
  Denisova-94,01

- Global regularity (small initial velocity):
  Denisova-07
  Saito, Shibata, X. Zhang-20 (low regularity)

- Sharp decay rates:
  Wang, Tice, Kim-14

Two-fluid interaction:

- Local existence:
  Denisova-94,01

- Global regularity (small initial velocity):
  Denisova-07
  Saito, Shibata, X. Zhang-20 (low regularity)

- Sharp decay rates:
  Wang, Tice, Kim-14

Two-fluid interaction without viscosity jump ($\mu = \mu_0 > 0$):

- Weak solutions:
  Simon-90
  Lions-96

Two-fluid interaction:

- Local existence:
  Denisova-94,01

- Global regularity (small initial velocity):
  Denisova-07
  Saito, Shibata, X. Zhang-20 (low regularity)

- Sharp decay rates:
  Wang, Tice, Kim-14

Two-fluid interaction without viscosity jump ($\mu = \mu_0 > 0$):

- Weak solutions:
  Simon-90
  Lions-96

- Low regularity positive density:
  Danchin, Mucha-12,13
  Paicu, P. Zhang, Z. Zhang-13

- Persistence of boundary regularity in 2D with $\rho > 0$:
  Liao, P. Zhang-16,19 ($C^{2+\gamma}$, $0 < \gamma < 1$)
  Danchin, X. Zhang-17 ($C^{1+\gamma}$ and smallness)
  G., García-Juárez-18 ($C^{1+\gamma}$ and $W^{2,\infty}$)

- Persistence of boundary regularity in 2D with $\rho > 0$:
  Liao, P. Zhang-16,19 ($C^{2+\gamma}$, $0 < \gamma < 1$)
  Danchin, X. Zhang-17 ($C^{1+\gamma}$ and smallness)
  G., García-Juárez-18 ($C^{1+\gamma}$ and $W^{2,\infty}$)

- Stokes–Navier-Stokes interaction (one fluid with density zero):
  Danchin, Mucha-19

- Persistence of boundary regularity in 2D with $\rho > 0$:
  Liao, P. Zhang-16,19 ($C^{2+\gamma}$, $0 < \gamma < 1$)
  Danchin, X. Zhang-17 ($C^{1+\gamma}$ and smallness)
  G., García-Juárez-18 ($C^{1+\gamma}$ and $W^{2,\infty}$)
- Stokes–Navier-Stokes interaction (one fluid with density zero): Danchin, Mucha-19

Small viscosity contrast in 2D:

- Paicu, P. Zhang-20 ($H^{5/2}$)
- **G.**, García-Juárez-21 ($C^{1+\gamma}$ and low regularity for the velocity)

Global regularity

Theorem

Let

- $\mathbb{R}^2 \supset D_0 \in C^{1+\gamma}$, $0 < \gamma < 1$
- $\rho_0, \mu_0 \in C^{\gamma}_{D_0}$ *with* $\rho_0 - \rho^\infty, \mu_0 - \mu^\infty \in L^2$
- $u_0 \in H^{\gamma+\varepsilon} \cap L^r$, $1 < r < 2$, $\nabla \cdot u_0 = 0$

Then, there exists $\delta > 0$ *such that if*

$$\left\|1 - \mu_0/\bar{\mu}\right\|_{L^\infty} \leq \delta, \quad \textit{with} \qquad \overline{\mu} = (\min \mu + \max \mu)/2,$$

there exists a unique global solution of N-SFB with

$$u \in C(\mathbb{R}^+; H^{\gamma+\varepsilon}) \cap L^1(\mathbb{R}^+; W^{1,\infty}) \cap L^1(\mathbb{R}^+; C^{1+\gamma}_{D(t)}),$$

$$D \in C(\mathbb{R}^+; C^{1+\gamma}),$$

where $D(t) = X(D_0, t)$*, with* X *the particle trajectories associated to the velocity field.*

Regularity of the velocity:

- In particular:

$$\|\sqrt{\rho}u\|_{L^2}^2(t) + \int_0^t \|\sqrt{\mu}\,\mathbb{D}u\|_{L^2}^2(\tau)d\tau \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2,$$

Regularity of the velocity:

- In particular:

$$\|\sqrt{\rho}u\|_{L^2}^2(t) + \int_0^t \|\sqrt{\mu}\,\mathbb{D}u\|_{L^2}^2(\tau)d\tau \leq \|\sqrt{\rho_0}u_0\|_{L^2}^2,$$

$$t^{1-\gamma-\varepsilon}\|\nabla u\|_{L^2}^2(t) + \int_0^t \tau^{1-\gamma-\varepsilon}\|\sqrt{\rho}D_t u\|_{L^2}^2(\tau)d\tau \leq C\|u_0\|_{\dot{H}^{\gamma+\varepsilon}}^2,$$

$$t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2(t) + \int_0^t \tau^{2-\gamma-\varepsilon}\|\nabla D_t u\|_{L^2}^2(\tau)d\tau \leq C\|u_0\|_{\dot{H}^{\gamma+\varepsilon}}^2,$$

with

$$D_t f = \partial_t f + u \cdot \nabla f.$$

Regularity of the velocity:

- In particular:

$$\|\sqrt{\rho}u\|_{L^2}^2(t) + \int_0^t \|\sqrt{\mu}\,\mathbb{D}u\|_{L^2}^2(\tau)d\tau \le \|\sqrt{\rho_0}u_0\|_{L^2}^2,$$

$$t^{1-\gamma-\varepsilon}\|\nabla u\|_{L^2}^2(t) + \int_0^t \tau^{1-\gamma-\varepsilon}\|\sqrt{\rho}D_t u\|_{L^2}^2(\tau)d\tau \le C\|u_0\|_{\dot{H}^{\gamma+\varepsilon}}^2,$$

$$t^{2-\gamma-\varepsilon}\|D_t u\|_{L^2}^2(t) + \int_0^t \tau^{2-\gamma-\varepsilon}\|\nabla D_t u\|_{L^2}^2(\tau)d\tau \le C\|u_0\|_{\dot{H}^{\gamma+\varepsilon}}^2,$$

with

$$D_t f = \partial_t f + u\cdot\nabla f.$$

- Critical initial regular velocity as for linear heat equation:

$$u_t^L - \Delta u^L = 0, \quad u^L(0) = u_0^L \in H^\gamma,$$

$$u^L \notin L^1(\mathbb{R}^+; H^{2+\gamma}) \subset L^1(\mathbb{R}^+; C^{1+\gamma}).$$

Ingredients:

- Step 1: $\sqrt{\rho}u \in L^\infty(0,T;L^2), \quad \sqrt{\mu}\mathbb{D}u \in L^2(0,T;L^2)$
  - Energy balance

Ingredients:

- Step 1: $\sqrt{\rho}u \in L^\infty(0, T; L^2)$, $\sqrt{\mu}\mathbb{D}u \in L^2(0, T; L^2)$
  - Energy balance
- Step 2: $t^{\frac{1-\gamma-\varepsilon}{2}}\nabla u \in L^\infty(0, T; L^2)$, $t^{\frac{1-\gamma-\varepsilon}{2}}D_t u \in L^2(0, T; L^2)$
  - Linearized system: $\rho(D_t v) = \nabla \cdot (\mu\mathbb{D}v - \mathbb{I}P)$
  - $D_t v \cdot$
  - $[D_t, \partial_j]f = -\partial_j u \cdot \nabla f$
  - $D_t \mu = 0$
  - Time weight: interpolation
  - $(\mathcal{H}^1)^* = BMO$
  - div-curl lemma
  - $\dot{H}^1 \subset BMO$

- Estimates:

$$\bar{\mu}\Delta v = \nabla \cdot (\mu \mathbb{D} v) - \nabla \cdot (\mu \mathbb{D} v - \bar{\mu} \mathbb{D} v)$$

- Estimates:

$$\bar{\mu}\Delta v = \nabla \cdot (\mu \mathbb{D} v) - \nabla \cdot (\mu \mathbb{D} v - \bar{\mu} \mathbb{D} v)$$

Then

$$\nabla v = \frac{1}{\bar{\mu}} \nabla \Delta^{-1} \mathbb{P} \nabla \cdot (\mu \mathbb{D} v) - \nabla \Delta^{-1} \mathbb{P} \nabla \cdot \left(\left(\frac{\mu}{\bar{\mu}} - 1 \right) \mathbb{D} v \right)$$

with $\mathbb{P} f = f - \nabla \Delta^{-1} \nabla \cdot f$.

- Estimates:

$$\bar{\mu}\Delta v = \nabla \cdot (\mu \mathbb{D} v) - \nabla \cdot (\mu \mathbb{D} v - \bar{\mu} \mathbb{D} v)$$

Then

$$\nabla v = \frac{1}{\bar{\mu}} \nabla \Delta^{-1} \mathbb{P} \nabla \cdot (\mu \mathbb{D} v) - \nabla \Delta^{-1} \mathbb{P} \nabla \cdot \left(\left(\frac{\mu}{\bar{\mu}} - 1 \right) \mathbb{D} v \right)$$

with $\mathbb{P} f = f - \nabla \Delta^{-1} \nabla \cdot f$. Furthermore

$$\|\nabla v\|_{L^p} \leq c(\delta) \|\nabla \Delta^{-1} \mathbb{P} \nabla \cdot (\mu \mathbb{D} v)\|_{L^p}, \qquad 2 \leq p \leq q < \infty$$

- Estimates:

$$\bar{\mu}\Delta v = \nabla\cdot(\mu\mathbb{D}v) - \nabla\cdot(\mu\mathbb{D}v - \bar{\mu}\mathbb{D}v)$$

Then

$$\nabla v = \frac{1}{\bar{\mu}}\nabla\Delta^{-1}\mathbb{P}\nabla\cdot(\mu\mathbb{D}v) - \nabla\Delta^{-1}\mathbb{P}\nabla\cdot\Big(\Big(\frac{\mu}{\bar{\mu}} - 1\Big)\mathbb{D}v\Big)$$

with $\mathbb{P}f = f - \nabla\Delta^{-1}\nabla\cdot f$. Furthermore

$$\|\nabla v\|_{L^p} \le c(\delta)\|\nabla\Delta^{-1}\mathbb{P}\nabla\cdot(\mu\mathbb{D}v)\|_{L^p}, \qquad 2\le p\le q<\infty$$

Gagliardo-Nirenberg

$$\|f\|_{L^p} \le c\|f\|_{L^2}^{\frac{2}{p}}\|\nabla f\|_{L^2}^{1-\frac{2}{p}},$$

and N-S equation yield

$$\|\nabla u\|_{L^p} \le c(\delta)\|\mu\mathbb{D}u\|_{L^2}^{\frac{2}{p}}\|\rho D_t u\|_{L^2}^{1-\frac{2}{p}}.$$

- Step 3: $t^{1-\frac{\gamma+\varepsilon}{2}} D_t u \in L^\infty(0, T; L^2), \quad t^{1-\frac{\gamma+\varepsilon}{2}} \nabla D_t u \in L^2(0, T; L^2)$
  - $$\frac{d}{dt}\left(t^{2-\gamma-\varepsilon} \int \rho |D_t u|^2 dx\right)$$
  - $$-t^{2-\gamma-\varepsilon} \int D_t u \cdot D_t \nabla P dx$$

- Step 3: $t^{1-\frac{\gamma+\varepsilon}{2}} D_t u \in L^\infty(0,T;L^2), \quad t^{1-\frac{\gamma+\varepsilon}{2}} \nabla D_t u \in L^2(0,T;L^2)$
  - $$\frac{d}{dt}\Big(t^{2-\gamma-\varepsilon}\int \rho|D_t u|^2 dx\Big)$$
  - $$-t^{2-\gamma-\varepsilon}\int D_t u \cdot D_t \nabla P dx$$
- Step 4: $\nabla u \in L^1(0,T;L^\infty)$

$$\nabla u = -\nabla\Delta^{-1}\mathbb{P}\nabla\cdot\big((\frac{\mu}{\bar\mu}-1)\mathbb{D}u\big) + \frac{1}{\bar\mu}\nabla\Delta^{-1}\mathbb{P}(\rho D_t u),$$

$$\nabla u = SIO\big((\frac{\mu}{\bar\mu}-1)\mathbb{D}u\big) + Regular$$

$\nabla u \in L^1(0, T; L^\infty)$

$$\nabla u = I + Regular,$$

where

$$I = \int_{D(t)} K(x-y)\Big(\frac{\mu^{in}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy$$
$$+ \int_{\mathbb{R}^2\setminus\overline{D(t)}} K(x-y)\Big(\frac{\mu^{out}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy$$

$\nabla u \in L^1(0, T; L^\infty)$

$$\nabla u = I + Regular,$$

where

$$I = \int_{D(t)} K(x-y)\Big(\frac{\mu^{in}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy + \int_{\mathbb{R}^2 \setminus \overline{D(t)}} K(x-y)\Big(\frac{\mu^{out}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy$$

- $\|\nabla u\|_{L^p}$ decay in time
- μ and ∇u in C^γ_D
- K even kernel

$\nabla u \in L^1(0, T; L^\infty)$

$$\nabla u = I + Regular,$$

where

$$I = \int_{D(t)} K(x-y)\Big(\frac{\mu^{in}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy + \int_{\mathbb{R}^2 \setminus \overline{D(t)}} K(x-y)\Big(\frac{\mu^{out}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy$$

- $\|\nabla u\|_{L^p}$ decay in time
- μ and ∇u in C^γ_D
- K even kernel

$$\int_0^t \|\nabla u\|_{L^\infty} d\tau \leq Control + c\delta \int_0^t \|\nabla u\|_{L^\infty} d\tau + c\delta \int_0^t \|\nabla u\|_{\dot{C}^\gamma_D} d\tau$$

- Step 5: $\nabla u \in L^1(0, T; \dot{C}^{\gamma}_D)$

$$\nabla u(x+h) - \nabla u(x) = J_D + J_{D^c} + Regular,$$

where

$$J_D = \int_{D(t)} \big(K(x+h-y) - K(x-y)\big)\Big(\frac{\mu^{in}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy$$

• Step 5: $\nabla u \in L^1(0, T; \dot{C}^{\gamma}_D)$

$$\nabla u(x+h) - \nabla u(x) = J_D + J_{D^c} + Regular,$$

where

$$J_D = \int_{D(t)} \big(K(x+h-y) - K(x-y)\big)\Big(\frac{\mu^{in}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy$$

- $SIO : \dot{C}^{\gamma} \to \dot{C}^{\gamma}$
- K even

• Step 5: $\nabla u \in L^1(0, T; \dot{C}^\gamma_D)$

$$\nabla u(x+h) - \nabla u(x) = J_D + J_{D^c} + Regular,$$

where

$$J_D = \int_{D(t)} \big(K(x+h-y) - K(x-y)\big)\Big(\frac{\mu^{in}(y)}{\bar{\mu}} - 1\Big)\mathbb{D}u(y)dy$$

- $SIO : \dot{C}^\gamma \to \dot{C}^\gamma$
- K even

$$J_D = L_1 + L_2 + L_3 + L_4 + L_5,$$

where

$$L_5 = \Big(\frac{\mu^{in}(x+h)}{\bar{\mu}} - 1\Big)\mathbb{D}u(x+h)\int_{D(t)} \big(K(x+h-y) - K(x-y)\big)dy.$$

$\nabla u \in L^1(0, T; \dot{C}^\gamma_D)$

$$|L_5| \leq c\delta \|\nabla u\|_{L^\infty} \mathcal{P}(\|D\|_{\mathrm{Lip}} + \|D\|_*) \|D\|_{\dot{C}^{1,\gamma}} \log\left(e + \|D\|_{\dot{C}^{1,\gamma}}\right) |h|^\gamma,$$

$\nabla u \in L^1(0, T; \dot{C}^\gamma_D)$

$$|L_5| \leq c\delta \|\nabla u\|_{L^\infty} \mathcal{P}(\|D\|_{\text{Lip}} + \|D\|_*) \|D\|_{\dot{C}^{1,\gamma}} \log\left(e + \|D\|_{\dot{C}^{1,\gamma}}\right) |h|^\gamma,$$

We can conclude:

$$\begin{aligned}\|\nabla u\|_{\dot{C}^\gamma_D} \leq & \frac{c\delta}{1 - c\delta} \|\nabla u\|_{L^\infty} \mathcal{P}(\|D\|_{\text{Lip}} + \|D\|_*) \|D\|_{\dot{C}^{1,\gamma}} \log\left(e + \|D\|_{\dot{C}^{1,\gamma}}\right) \\ & + \frac{c}{1 - c\delta} \|\mu\|_{\dot{C}^\gamma_D} \|\nabla u\|_{L^\infty} + Control.\end{aligned}$$

$\nabla u \in L^1(0, T; \dot{C}^\gamma_D)$

$$|L_5| \leq c\delta \|\nabla u\|_{L^\infty} \mathcal{P}(\|D\|_{\text{Lip}} + \|D\|_*) \|D\|_{\dot{C}^{1,\gamma}} \log\left(e + \|D\|_{\dot{C}^{1,\gamma}}\right) |h|^\gamma,$$

We can conclude:

$$\begin{aligned}\|\nabla u\|_{\dot{C}^\gamma_D} \leq & \frac{c\delta}{1 - c\delta} \|\nabla u\|_{L^\infty} \mathcal{P}(\|D\|_{\text{Lip}} + \|D\|_*) \|D\|_{\dot{C}^{1,\gamma}} \log\left(e + \|D\|_{\dot{C}^{1,\gamma}}\right) \\ & + \frac{c}{1 - c\delta} \|\mu\|_{\dot{C}^\gamma_D} \|\nabla u\|_{L^\infty} + \mathit{Control}.\end{aligned}$$

- Step 6: Closing all estimates:

$$\begin{aligned}\frac{d}{dt} \|D\|_{Lip} &\leq \|\nabla u\|_{L^\infty} \|D\|_{Lip}, \\ \frac{d}{dt} \|D\|_{\dot{C}^{1+\gamma}_D} &\leq \|\nabla u\|_{L^\infty} \|D\|_{\dot{C}^{1+\gamma}_D} + \|D\|_{Lip}^{1+\gamma} \|\nabla u\|_{\dot{C}^\gamma_D},\end{aligned}$$

Closing all estimates:

For

$$y(t) = \int_0^t \|\nabla u\|_{L^\infty}(\tau) d\tau,$$

it is possible to get

$$y(T) \leq \delta C_1 e^{C_2(1+y(T))e^{c_3 \delta e^{c_4 y(T)}}} + C_5.$$

Closing all estimates:

For

$$y(t) = \int_0^t \|\nabla u\|_{L^\infty}(\tau)d\tau,$$

it is possible to get

$$y(T) \leq \delta C_1 e^{C_2(1+y(T))e^{c_3\delta e^{c_4 y(T)}}} + C_5.$$

Assume that

$$\delta \leq \min\{\frac{e^{-2c_4C_5}}{c_3}, \frac{C_5 e^{-eC_2(1+2C_5)}}{2C_1}\}.$$

then

$$y(T) = \int_0^T \|\nabla u\|_{L^\infty} dt < 2C_5.$$

Thank you!