Nonlocal pulling in reaction–diffusion equations

CY Days in Nonlinear Analysis

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Introduction

Reaction–diffusion equations

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } (0,+\infty)\times\mathbb{R}^n \\ u(0,\cdot) = u_0 \geq 0 \end{cases}$$

u_0 and f regular, $f(0)=0$, $f(u)<0$ if $u>1$

Basic properties

Consequences of the parabolic comparison principle: nonnegativity, well-posedness, **spreading** depending on f, u_0

Spreading speeds

Definition

Minimal spreading speed:

$$\underline{c^\star} = \sup \left\{ c \geq 0 \mid \lim_{t\to+\infty} \inf_{0\leq|x|\leq ct} u(t,x) > 0 \right\}$$

Maximal spreading speed:

$$\overline{c^\star} = \inf \left\{ c \geq 0 \mid \lim_{t\to+\infty} \sup_{ct\leq|x|} u(t,x) = 0 \right\}$$

$[\underline{c^\star}, \overline{c^\star}] \subset [0, +\infty]$ depends on u_0

From compactly supported initial data

If u_0 compactly supported, then by comparison with a linear parabolic equation, $\overline{c^\star} < +\infty$

Numerical observation for many natural f

Convergence $u(t,x) \to U(|x| - c^\star t + o(t))$ to a planar traveling wave with speed $c^\star = \underline{c^\star} = \overline{c^\star}$ and profile U satisfying $U(+\infty) = 0$ and $\liminf_{-\infty} U > 0$

Isotropy of $c^\star$: from now on, restriction to 1D

Pulled fronts vs. pushed fronts

Stokes, *Math. Bio.*, 1976

Definition: the linearly determined speed c_{lin}

Smallest nonnegative real number $c \geq 0$ such that the stationary linearized equation at $u \simeq 0$ in the moving frame $x - ct$,

$$-u'' - cu' = f'(0)u,$$

admits positive solutions decaying to 0 at $+\infty$

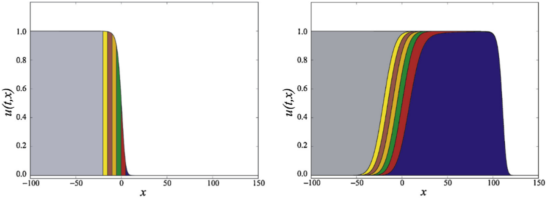
Definition: pulled and pushed fronts

The front $u(t,x)$ with spreading speed $c^\star$ emanating from u_0 compactly supported is:

- pulled if $c^\star = c_{\text{lin}}$
- pushed otherwise (and then $c^\star > c_{\text{lin}}$)

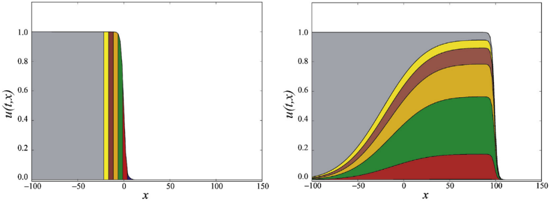
Equivalent definition: inside dynamics

Garnier, Giletti, Hamel, Roques, *JMPA*, 2012



(a) $t = 0$ (b) $t = 60$

Figure: Pulled fronts: only leading individuals matter



(a) $t = 0$ (b) $t = 700$

Figure: Pushed fronts: all individuals matter

Example: the Fisher–KPP equation

Fisher, 1937; Kolmogorov, Petrovskii, Piskunov, 1937

$$\partial_t u - d\partial_{xx} u = ru(1-u)$$

Model in population genetics, population dynamics
$d, r > 0$ (without loss of generality $r = d = 1$ possible)
ODE: $u = 0$ unstable, $u = 1$ stable

Theorem: spreading, pulling, convergence

$c^\star = c_{\text{lin}} = 2\sqrt{rd}$ and moreover

$$\lim_{t\to+\infty} \sup_{|x|<(2\sqrt{rd}-\varepsilon)t} |1-u(t,x)| = 0$$

Example: the Fisher–KPP equation

Finding the linearly determined speed

Linearization at $u \simeq 0$ in the moving frame $z = x - ct$, $c \geq 0$:

$$\partial_t u - d\partial_{zz} u - c\partial_z u = ru$$

Stationary problem: $-du'' - cu' = ru$

Exponential ansatz: $u(z) = \exp(\mu z)$, $\mu \in \mathbb{C}$

Dispersion relation: $d\mu^2 + c\mu + r = 0$

$\mu_\pm \in \mathbb{R}$ iff $c \geq 2\sqrt{rd}$ and then $\mu_\pm < 0$: $c_{\text{lin}} = 2\sqrt{rd}$

Example: the Fisher–KPP equation

Solving the resulting problem

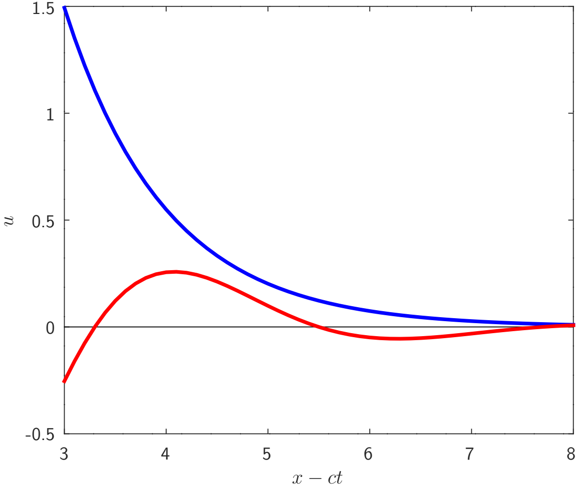


Figure: Solutions of $-u'' - cu' = u$ for $c = 2$ (blue), $c = 1.4$ (red)

Example: the Fisher–KPP equation

Building super- and sub-solution candidates

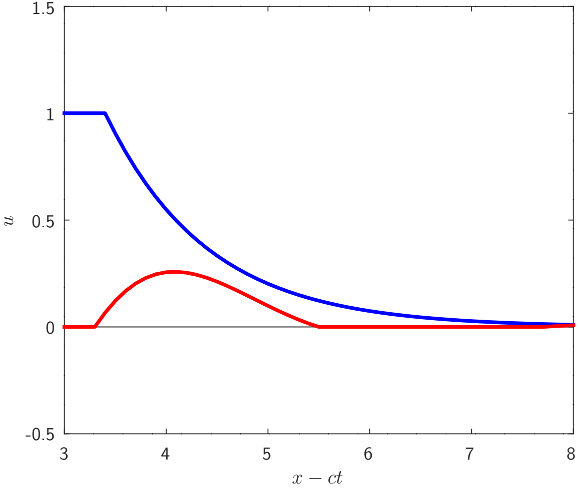


Figure: Super- and sub-solution candidates (blue and red resp.)

Example: the Fisher–KPP equation

Validating the super- and sub-solution candidates

Validation of the super-solution moving at speed $c = 2\sqrt{rd}$:

$$\partial_t \overline{u} - d\partial_{xx}\overline{u} = r\overline{u} \geq r\overline{u}(1-\overline{u})$$

$\overline{u} \geq u$ at $t = 0$: up to changing 1 by $\max(\max(u_0), 1)$

Example: the Fisher–KPP equation

Validating the super- and sub-solution candidates

Validation of the super-solution moving at speed $c = 2\sqrt{rd}$:

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$\overline{u} \geq u$ at $t=0$: up to changing 1 by $\max(\max(u_0), 1)$

Validation of a perturbed sub-solution moving at speed $c \in (0, 2\sqrt{rd})$ with a small parameter $\delta > 0$ such that $c^2 - 4(1-\delta)rd < 0$ remains true:

$$\partial_t \underline{u} - d\partial_{xx}\underline{u} = (1-\delta)r\underline{u} \leq r\underline{u}(1-\underline{u}) \quad \text{provided } \underline{u} \leq \delta$$

$\underline{u} \leq u$ at $t=1$: up to decreasing δ again

Example: the Fisher–KPP equation

Using the super- and sub-solutions as barriers

From the super-solution with $c = 2\sqrt{rd}$:

$$\forall c' > 2\sqrt{rd} \quad \lim_{t\to+\infty} \sup_{|x|>c't} u(t,x) = 0 \implies \overline{c^\star} \leq 2\sqrt{rd}$$

From the family of sub-solutions with $c < 2\sqrt{rd}$, $c \simeq 2\sqrt{rd}$:

$$\forall c' \in (0, 2\sqrt{rd}) \quad \liminf_{t\to+\infty} \inf_{|x|<c't} u(t,x) > 0 \implies \underline{c^\star} \geq 2\sqrt{rd}$$

Convergence to 1 in $\{|x| < (2\sqrt{rd} - \varepsilon)t\}$:
Liouville-type result on uniformly positive entire solutions

Pushed examples

Monostable equation with strong convexity at the origin

$$\partial_t u - \partial_{xx} u = u(u+\alpha)(1-u), \quad \alpha \in \left[0, \frac{1}{2}\right)$$

$c_{\text{lin}} = 2\sqrt{\alpha}$ but $c^\star = \frac{\sqrt{2}(1+2\alpha)}{2} > 2\sqrt{\alpha}$

Proof failure: when validating the super-solution

Bistable equation

$$\partial_t u - \partial_{xx} u = u(1-u)(u-\theta), \quad \theta \in \left(0, \frac{1}{2}\right)$$

$c_{\text{lin}} = 0$ but $c^\star = \frac{\sqrt{2}(1-2\theta)}{2}$ (large u_0) or extinction (small u_0)

Proof failure: when constructing the sub-solution and when validating the super-solution

Extension to more general media?

Heterogeneous Fisher–KPP equation

$$\partial_t u - \partial_{xx} u = f(u,t,x)$$

with assumptions on f generalizing $f(u) = u(1-u)$; for simplicity, focus on

$$f(u,t,x) = r(t,x)u(1-u) \quad \text{or} \quad u(r(t,x)-u)$$

The sign of r matters (a lot)

- Negative r: extinction, no spreading
- Positive r: spreading, no extinction
- Sign-changing r: case-by-case

Focus on spreading properties: from now on, $\inf_{(t,x)\in(0,+\infty)\times\mathbb{R}} r(t,x) > 0$

Positive heterogeneous environments

By comparison, $0 < 2\sqrt{\inf r} \leq \underline{c^\star} \leq \overline{c^\star} \leq 2\sqrt{\sup r} < +\infty$

Toward a generalization of the homogeneous Fisher–KPP result

- Equality $\underline{c^\star} = \overline{c^\star}$? Estimates?
- Definition and calculation of c_{lin}?
- Equivalence of the two definitions of pulled & pushed?
- If not, other regimes?

The easiest case

Berestycki, Hamel, Nadin, *J. Func. Anal.*, 2008

Confined heterogeneities

$r(t,x)$ independent of (t,x) if $|x| > R$ or $t > T$

$c^\star = 2\sqrt{r(T+1,R+1)}$ and with minimal adaptation:

- pulled in the sense of Stokes
- pulled in the sense of Garnier *et al.*

Only leading individuals matter, and leading individuals only feel the asymptotic growth rate

A complicated case

Garnier, Giletti, Nadin, *JDDE*, 2012

An environment oscillating slower and slower

$r(t,x) = r(x) = R(\phi(x))$ with R periodic, $\phi' > 0$,
$\lim_{x\to+\infty} \phi(x) = +\infty$, $\lim_{x\to+\infty} x\phi'(x) = 0$

Oscillations of the rightward spreading speed:

$$\underline{c^\star} = 2\sqrt{\min R} < \overline{c^\star} = 2\sqrt{\max R}$$

Pulled? Pushed?

An important class

Environmental change with constant speed

$r(x - c_{\text{het}}t)$ with $c_{\text{het}} \geq 0$

Arise naturally in:

- climate change models
- river models
- systems of reaction–diffusion equations

Analysis in a moving frame... but which one?

- $x - c_{\text{het}}t$ (stationary medium)?
- $x - c^{\star}t$ (stationary wave)?

Positive shifting environment: known results

Flavor and intuition: the simplest case

The simplest example of shifting medium

Piecewise-constant shifting medium with one jump

$$f(u,t,x) = r(x - c_{\text{het}}t)u(1-u)$$

$$r = r_1\mathbf{1}_{(-\infty,0)} + r_2\mathbf{1}_{[0,+\infty)}, \quad c_{\text{het}} \geq 0$$

Expectation:

- $c^\star_{\text{left}} = 2\sqrt{r_1}$
- $c^\star_{\text{right}} = 2\sqrt{r_2}$ if c_{het} small – how small?
- $c^\star_{\text{right}} = 2\sqrt{r_1}$ if c_{het} large – how large?

And in between?

The decreasing case

Theorem: if $r_1 > r_2$, locking occurs in between

If $r_1 > r_2$, $c^\star_{\text{left}} = 2\sqrt{r_1}$ and

$$c^\star_{\text{right}} = \begin{cases} 2\sqrt{r_2} & \text{if } c_{\text{het}} < 2\sqrt{r_2} \\ 2\sqrt{r_1} & \text{if } c_{\text{het}} > 2\sqrt{r_1} \\ c_{\text{het}} & \text{if } c_{\text{het}} \in [2\sqrt{r_2}, 2\sqrt{r_1}] \end{cases}$$

Locking: invasion front located at the environmental heterogeneity

Illustration: a locked front

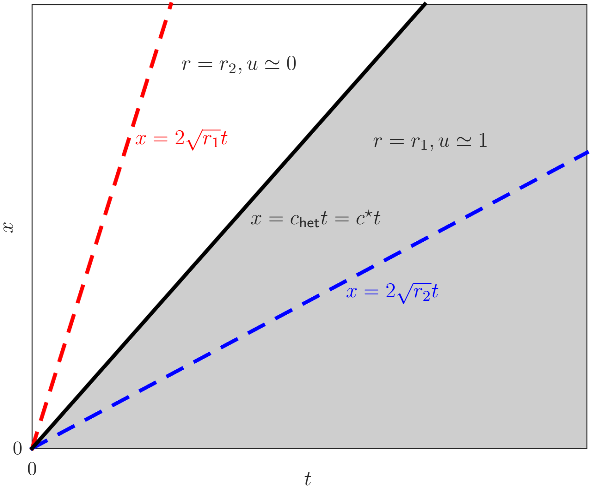

Figure: Spreading in (t,x)-plane ($r_1 = 4$, $r_2 = 1/9$, $c_{\text{het}} = \sqrt{2}$)

Illustration: pulling–locking–pulling

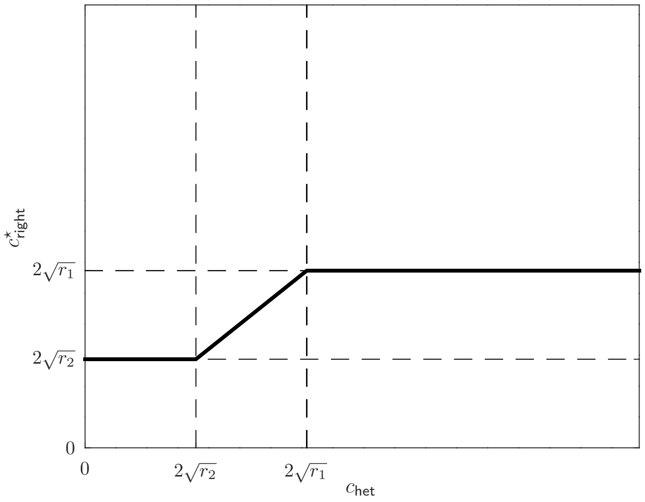

Figure: The spreading speed as function of the environmental speed ($r_1 = 4$, $r_2 = 1$)

The increasing case

Theorem: if $r_1 < r_2$, nonlocal pulling occurs in between

If $r_1 < r_2$, $c^\star_{\text{left}} = 2\sqrt{r_1}$ and

$$c^\star_{\text{right}} = \begin{cases} 2\sqrt{r_2} & \text{if } c_{\text{het}} < 2\sqrt{r_2} \\ 2\sqrt{r_1} & \text{if } c_{\text{het}} > 2\sqrt{r_1} + 2\sqrt{r_2 - r_1} \\ F(c_{\text{het}}) & \text{if } c_{\text{het}} \in [2\sqrt{r_2}, 2\sqrt{r_1} + 2\sqrt{r_2 - r_1}] \end{cases}$$

$$\text{with} \quad F(c_{\text{het}}) = \frac{c_{\text{het}} - 2\sqrt{r_2 - r_1}}{2} + \frac{2r_1}{c_{\text{het}} - 2\sqrt{r_2 - r_1}}$$

Nonlocal pulling: invasion front slower than the environmental heterogeneity, so $r = r_1$ around the front, but still $c^\star > 2\sqrt{r_1}$ due to the advantageous exponential tail ahead of the heterogeneity

Illustration: a nonlocally pulled front

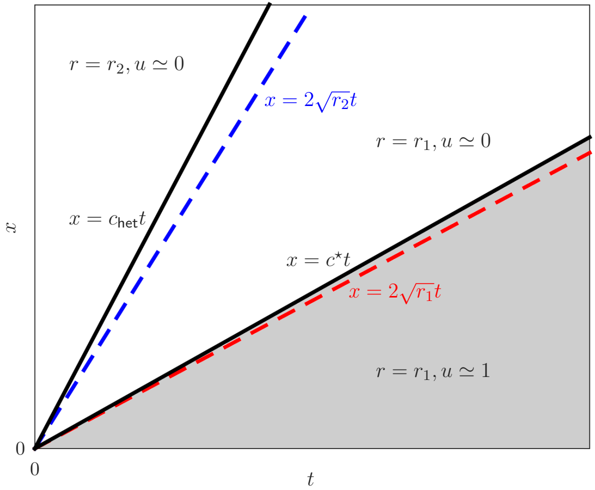

Figure: Spreading in (t, x)-plane ($r_1 = 1/9$, $r_2 = 1$, $c_{\text{het}} \simeq 2.37$)

Illustration: pulling–nonlocal pulling–pulling

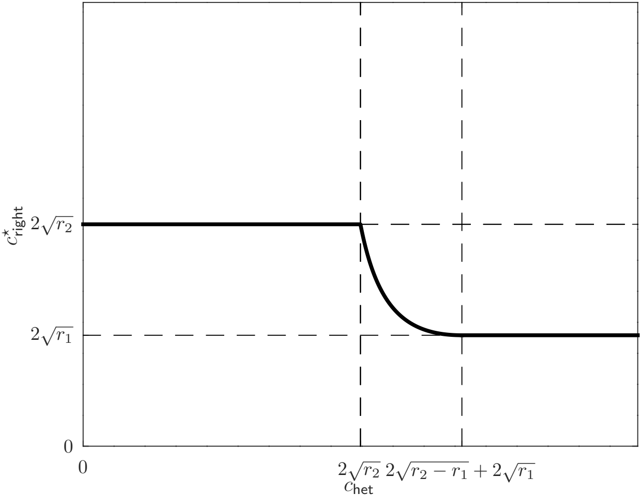

Figure: The spreading speed as function of the environmental speed ($r_1 = 1$, $r_2 = 4$)

Relation with pushed and pulled fronts

Expectations/conjectures

Locked fronts:

- pushed in the sense of Garnier *et al.*
- but pulled(*ish*) in the sense of Stokes

Nonlocally pulled fronts:

- pulled in the sense of Garnier *et al.*
- but pushed(*ish*) in the sense of Stokes

Relation with pushed and pulled fronts

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- but pulled(*ish*) in the sense of Stokes

Nonlocally pulled fronts:

- pulled in the sense of Garnier *et al.*
- but pushed(*ish*) in the sense of Stokes

"Whatever the case may be – and we admit that insisting on such a classification is somewhat pedantic – we see that novel modes of invasion exist"
– M. Holzer, A. Scheel, 2014

How to predict $c^\star = F(c_{\text{het}})$ (1/2)

Change of variable $x = y + c_{\text{het}} t$:

$$\partial_t u - \partial_{yy} u - c_{\text{het}} \partial_y u = r(y) u(1-u)$$

$$r = r_1 \mathbf{1}_{(-\infty,0)} + r_2 \mathbf{1}_{[0,+\infty)}$$

Educated guess in the wake of the heterogeneity $(y < 0)$

u converges to a traveling wave with speed $c^\star > 2\sqrt{r_1}$, that decays like $\mathrm{e}^{-\mu(y-(c^\star - c_{\text{het}})t)}$ with $\mu = \frac{1}{2}\left(c^\star - \sqrt{(c^\star)^2 - 4r_1}\right)$ solution of $\mu^2 - c^\star \mu + r_1 = 0$

Educated guess ahead of the heterogeneity $(y > 0)$

Since $c^\star < c_{\text{het}}$, $u^2 \ll u$, whence u behaves like $\mathrm{e}^{-\left(\frac{y^2}{4t} + \frac{c_{\text{het}}}{2} y - \frac{4r_2 - c_{\text{het}}^2}{4} t\right) + o(t)}$

How to predict $c^\star = F(c_{\text{het}})$ (2/2)

Matching asymptotics at $y = 0$:

$$\left(\frac{c_{\text{het}}}{2}\right)^2 - 2\mu\frac{c_{\text{het}}}{2} + \mu c^\star - r_2 = 0$$

$$\implies \frac{c_{\text{het}}}{2} = \mu \pm \sqrt{\mu^2 - \mu c^\star + r_2} = \mu \pm \sqrt{r_2 - r_1}$$

Continuity of $c^\star(c_{\text{het}})$ at $c_{\text{het}} = 2\sqrt{r_2}$:

$$c^\star(c_{\text{het}}) = 2\sqrt{r_2} \implies \mu = \sqrt{r_2} - \sqrt{r_2 - r_1}$$

Inversion of $\mu(c^\star) = \frac{c_{\text{het}}}{2} - \sqrt{r_2 - r_1}$:

$$\mu(c^\star)^2 - c^\star\mu(c^\star) + r_1 = 0 \iff c^\star(\mu) = \mu + \frac{r_1}{\mu}$$

Conclusion

$$c^\star = F(c_{\text{het}}) = \frac{c_{\text{het}} - 2\sqrt{r_2 - r_1}}{2} + \frac{2r_1}{c_{\text{het}} - 2\sqrt{r_2 - r_1}}$$

Literature

First heuristics

Venegas-Ortiz, Allen, Evans, *Genetics*, 2014

Model for horizontally transmitted hitchhiking traits

$$\begin{cases} \partial_t u = \partial_{xx} u + u(1-u-v) - \beta u + \gamma uv & \text{(carriers)} \\ \partial_t v = \partial_{xx} v + v(1-u-v) + \beta u - \gamma uv & \text{(non-carriers)} \end{cases}$$

Transformation $w = u + v$:

$$\begin{cases} \partial_t u = \partial_{xx} u + u(1-\beta-\gamma u-(1-\gamma)w) \\ \partial_t w = \partial_{xx} w + w(1-w) \end{cases}$$

- $c_w^\star = 2$
- $\{x \gg 2t\}$: u feels $r_2 = 1-\beta > 0$
- $\{x \ll 2t\}$: u feels $r_1 = \gamma-\beta > 0$
- If $\beta < \gamma < 1$, nonlocal pulling of u predicted

Incorrect result for $c_u^\star$ (wrong ansatz ahead of $x = 2t$)

First rigorous analysis

Holzer, Scheel, *SIAM J. Math. Anal.*, 2014

Triangular ad-hoc system

$$\begin{cases} \partial_t u = \partial_{xx} u + u(1-u) \\ \partial_t v = d\partial_{xx} v + g(u)v - v^3 \end{cases}$$

- $c_u^\star = 2$
- $g_{|[0,1]} > 0$, $g'(1) < 0$, $2\sqrt{dg(1)}, 2\sqrt{dg(0)} < 2$
- $c_v^\star = 2$ locked if some principal eigenvalue positive
  Proof: dynamical system approach
- Nonlocal pulling of v if principal eigenvalue in an interval $(\lambda_{\text{crit}}, 0)$
  Proof: super-sub-solution (cooperative system)
- Claim: $c_v^\star = 2\sqrt{dg(1)}$ locally pulled if principal eigenvalue smaller than λ_{crit}

First exhibition in a classical model

Girardin, Lam, *Proc. of the London Math. Soc.*, 2019

2-species Lotka–Volterra competition–diffusion system

$$\begin{cases} \partial_t u = \partial_{xx} u + u(1-u-av) \\ \partial_t v = d\partial_{xx} v + rv(1-v-bu) \end{cases}$$

- $0 < a < 1 < b$: monostable strong–weak competition
- $2\sqrt{rd} > 2$: v weaker competitor but faster spreader
- $c_v^\star = 2\sqrt{rd}$ (coupling: not trivial)
- Local front for u pushed or pulled (Lewis *et al.*, *J. Math. Biol.*, 2002)
- Nonlocal pulling of u iff local front slower
  Proof: super-sub-solution for 2-species competitive systems

First exhibition in a classical model

Girardin, Lam, *Proc. of the London Math. Soc.*, 2019

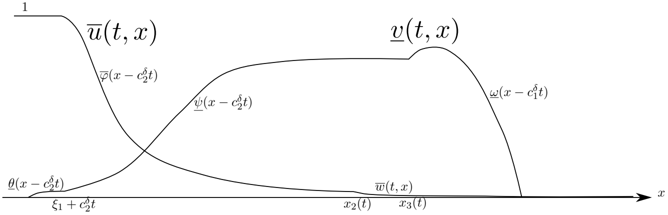

Figure: Complicated super-solution

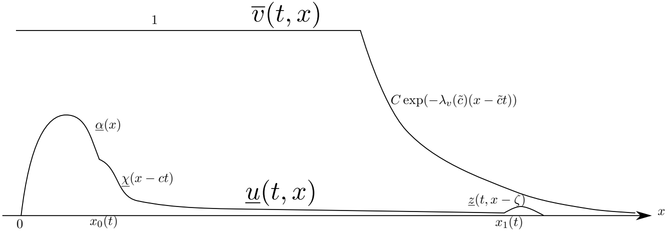

Figure: Complicated sub-solution

Since 2019

- WKB–Hamilton–Jacobi approach for 2-species competitive systems: Liu, Liu, Lam, *DCDS-A*, 2020
- Predator-prey system with 2 predators and 1 prey: Ducrot, Giletti, Guo, Shimojo, *Nonlinearity*, 2020
- Partial results for 3-species competitive systems: Liu, Liu, Lam, *JDE*, 2021
- Single equation with shifting diffusivity: Faye, Giletti, Holzer, *DCDS-S*, 2021
- WKB–Hamilton–Jacobi approach for general scalar equations with shifting growth rate: Lam, Yu, preprint, 2021
- SIR system with arbitrarily many spreading epidemics: Ducasse, Nordmann, in preparation

Methods of proof

The two known methods and their limitations

Super-sub-solution construction

- Comparison principle required
- Complicated constructions

WKB–Hamilton–Jacobi approach

- Cannot deal with areas of sub-linear size that might increase nonlocal pulling
- Cannot deal with locally pushed fronts

A nonlocally pulling patch

Work in progress with T. Giletti, H. Matano

A not-so-simple shifting medium

Piecewise-constant shifting medium with two jumps and a higher central patch

$$f(u,t,x) = r(x - c_{\text{het}}t)u(1-u) \quad \text{or} \quad u(r(x - c_{\text{het}}t) - u)$$

$$r = r_1 \mathbf{1}_{(-\infty,0)} + r_2 \mathbf{1}_{[0,L)} + r_3 \mathbf{1}_{[L,+\infty)}, \quad c_{\text{het}} \geq 0, \quad L > 0$$

Higher central patch: $r_2 > \max(r_1, r_3)$

Cannot be analyzed by WKB–Hamilton–Jacobi approach

Expectation (focusing on rightward spreading):

- $c^\star = 2\sqrt{r_3}$ if c_{het} small – how small?
- $c^\star = 2\sqrt{r_1}$ if c_{het} large – how large?

And in between? Impact of r_2, L?

Results

Two important quantities

Critical length $\underline{L}$

$$\underline{L} = \begin{cases} 0 & \text{if } r_1 = r_3 \\ \frac{1}{\sqrt{r_2 - \max(r_1, r_3)}} \operatorname{arccot}\left(\sqrt{\frac{r_2 - \max(r_1,r_3)}{|r_1 - r_3|}}\right) & \text{if } r_1 \neq r_3 \end{cases}$$

where $\operatorname{arccot} = \left(\cot_{|(0,\pi)}\right)^{-1}$

Generalized principal eigenvalue λ_1

1. If $r_1 > r_3$ and $L \leq \underline{L}$, $\lambda_1 = -r_1$;
2. If $r_1 < r_3$ and $L \leq \underline{L}$, $\lambda_1 = -r_3$;
3. If $r_1 = r_3$ or $L > \underline{L}$, λ_1 unique solution in $\left(-r_2, \min\left(-\max(r_1, r_3), \frac{\pi^2}{L^2} - r_2\right)\right)$ of

$$\cot(L\sqrt{r_2 + \lambda_1}) = \frac{r_2 + \lambda_1 - \sqrt{(r_1 + \lambda_1)(r_3 + \lambda_1)}}{\sqrt{r_2 + \lambda_1}(\sqrt{-r_1 - \lambda_1} + \sqrt{-r_3 - \lambda_1})}$$

Main result

Theorem: locking and nonlocal pulling occur in between

$$c^\star = \begin{cases} 2\sqrt{r_3} & \text{if } c_{\text{het}} < 2\sqrt{r_3} \\ c_{\text{het}} & \text{if } 2\sqrt{r_3} \leq c_{\text{het}} \leq 2\sqrt{-\lambda_1} \\ F(c_{\text{het}}) & \text{if } 2\sqrt{-\lambda_1} < c_{\text{het}} < 2\sqrt{-\lambda_1 - r_1} + 2\sqrt{r_1} \\ 2\sqrt{r_1} & \text{if } 2\sqrt{-\lambda_1 - r_1} + 2\sqrt{r_1} \leq c_{\text{het}} \end{cases}$$

with $F(c_{\text{het}}) = \dfrac{c_{\text{het}} - 2\sqrt{-\lambda_1 - r_1}}{2} + \dfrac{2r_1}{c_{\text{het}} - 2\sqrt{-\lambda_1 - r_1}}$

Illustration: case $r_3 < r_1$ and $L \leq \underline{L}$

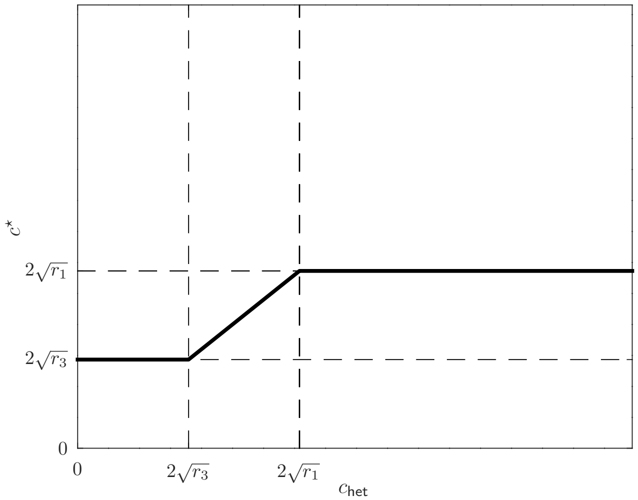

Figure: The spreading speed as function of the environmental speed ($r_1 = 4$, $r_3 = 1$, $r_2 = 9$, $\lambda_1 = -4$)

Illustration: case $r_3 < r_1$ and $L > \underline{L}$

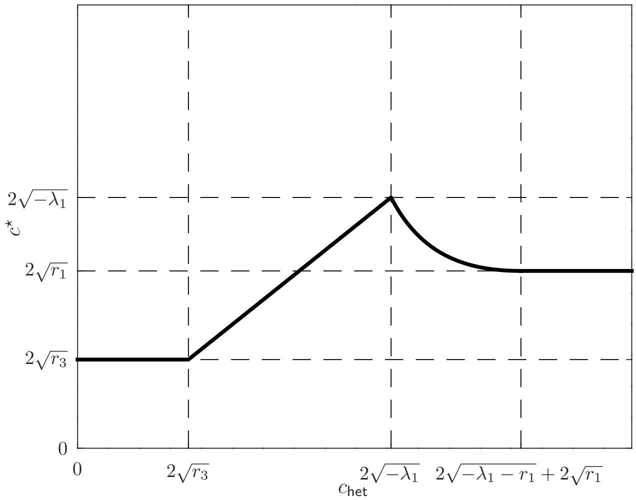

Figure: The spreading speed as function of the environmental speed ($r_1 = 4$, $r_3 = 1$, $r_2 = 9$, $\lambda_1 = -8$)

Illustration: case $r_1 < r_3$ and $L \leq \underline{L}$

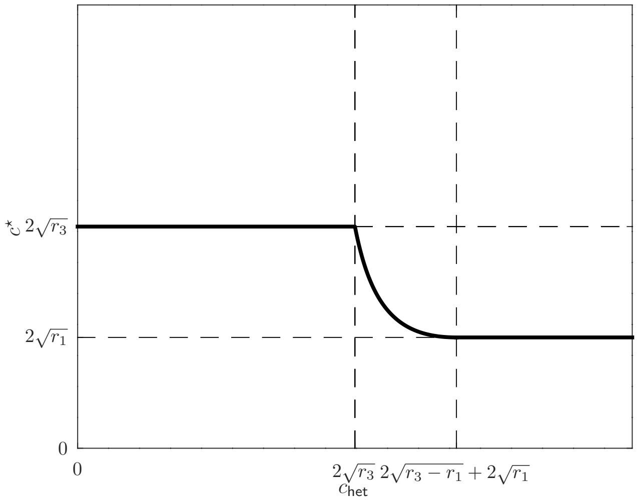

Figure: The spreading speed as function of the environmental speed ($r_1 = 1$, $r_3 = 4$, $r_2 = 9$, $\lambda_1 = -4$)

Illustration: case $r_1 < r_3$ and $L > \underline{L}$

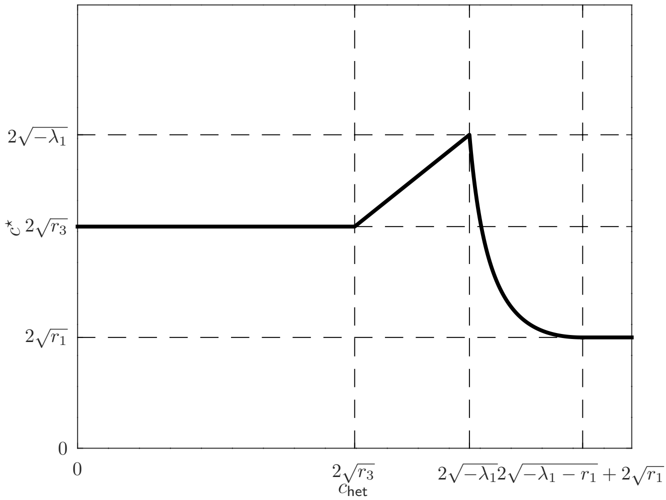

Figure: The spreading speed as function of the environmental speed ($r_1 = 1$, $r_3 = 4$, $r_2 = 9$, $\lambda_1 = -8$)

Illustration: case $r_1 = r_3$

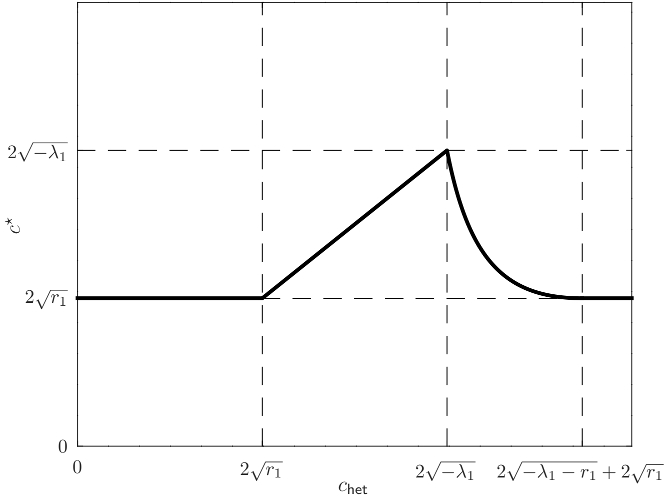

Figure: The spreading speed as function of the environmental speed ($r_1 = r_3 = 1$, $r_2 = 9$, $\lambda_1 = -4$)

Dependency of the spreading speed on other parameters

- $c^\star$ as function of λ_1: explicit closed-form formula, Lipschitz-continuous (only)
- $c^\star$ or λ_1 as functions of L or r_2: monotonic and continuous but otherwise implicit

Illustration

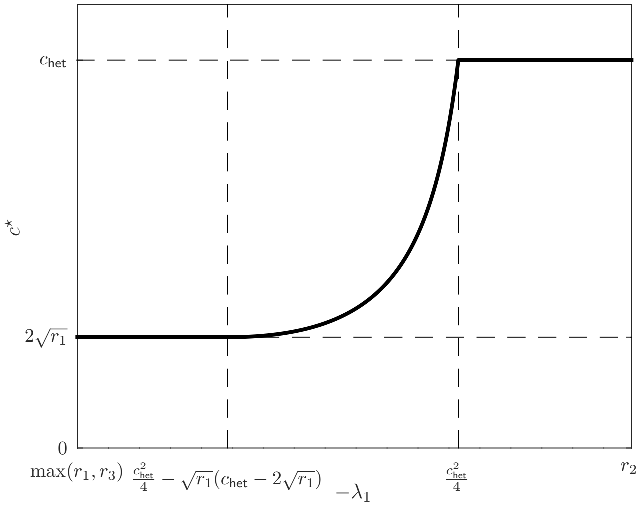

Figure: The spreading speed as function of the generalized principal eigenvalue ($r_1 = 1$, $r_2 = 16$, $r_3 = 4$, $c_{\text{het}} = 7$)

Sketch of proof

In the moving frame $x - c_{\text{het}}t$

Change of variable:

$$v(t,y) = u\left(t, Ly + c_{\text{het}}t\right) \mathrm{e}^{\frac{c_{\text{het}}^2 t}{4} + \frac{c_{\text{het}} Ly}{2}}$$

Linearization at $v \simeq 0$:

$$\partial_t v - \frac{1}{L^2}\partial_{yy} v = m(y) v$$

with $m(y) = r_1 \mathbf{1}_{y<0} + r_2 \mathbf{1}_{0\leq y<1} + r_3 \mathbf{1}_{1\leq y}$

Reminder

When $r_2 = r_3$, $v(t,y) \sim \mathrm{e}^{r_2 t}$

Educated guess

$v(t,y) \sim \mathrm{e}^{-\lambda_1 t}\varphi_1(y)$ where (λ_1, φ_1) principal eigenpair of $-\mathcal{L} = -L^{-2}\partial_{yy} - m$

Generalized principal eigenproblem

Berestycki, Rossi, *CPAM*, 2015

$$\begin{cases} -\mathcal{L}\varphi = \lambda\varphi & \text{in } \mathbb{R} \\ \varphi > 0 & \text{in } \mathbb{R} \end{cases}$$

Krein–Rutman-type uniqueness despite the compactness default?

Generalized principal eigenproblem

Berestycki, Rossi, *CPAM*, 2015

$$\begin{cases} -\mathcal{L}\varphi = \lambda\varphi & \text{in } \mathbb{R} \\ \varphi > 0 & \text{in } \mathbb{R} \end{cases}$$

Krein–Rutman-type uniqueness despite the compactness default?

No: the set of generalized principal eigenvalues has the form $(-\infty, \lambda_1]$

Generalized principal eigenproblem

Characterizations of λ_1:

$$\begin{aligned}\lambda_1 &= \sup\left\{\lambda \in \mathbb{R} \mid \exists \varphi > 0 \ \ -\mathcal{L}\varphi \geq \lambda\varphi\right\} \\ &= \lim_{R\to+\infty} \lambda_{1,\mathsf{Dir}}(-\mathcal{L}, B(0,R))\end{aligned}$$

Properties

For $\mathcal{L} = L^{-2}\partial_{yy} + m$,

1. $\lambda_1 \in [-r_2, -\max(r_1, r_3)]$
2. given (λ, φ), if φ bounded or if $\lambda = -\max(r_1, r_3)$, then $\lambda = \lambda_1$
3. in both cases, explicit construction of (λ_1, φ_1) ($\mathcal{C}^1$ regularity)

Super-sub-solution

Construction in the nonlocally pulled regime

Gluing KPP semi-linear super-sub-solutions in $\{y < 0\}$ and rescaled linear super-sub-solutions in $\{y > 0\}$

Validation in the nonlocally pulled regime

- Condition for super-solution in $\{y > 0\}$: $c > F(c_{\text{het}})$
- Condition for sub-solution in $\{y > 0\}$: $c < F(c_{\text{het}})$
- Angle conditions (super-solution $\wedge$, sub-solution $\vee$) at $y = 0$: redundant

Outside of the nonlocally pulled regime: more standard super-sub-solutions

Conclusion

Summary

- Pulled vs. pushed dichotomy in homogeneous media
- New regimes in shifting media (climate change, rivers, systems): locked, nonlocally pulled
- Nonlocal pulling: active research topic since 2014
- Explicit formula for the nonlocally pulled speed
- 2 methods of proof with pros & cons

Perspectives — The end

- Shifting patch with time-dependent length and speed
- Smooth variations of the growth rate
- Inside dynamics of locked and nonlocally pulled fronts
- Position of level sets $X(t) = c^\star t + o(t) = \ldots ?$
- Long-term goal: application to systems

Reaction–diffusion equations

$$\begin{cases} \partial_t u - \Delta u = f(u) & \text{in } (0,+\infty) \times \mathbb{R}^n \\ u(0,\cdot) = u_0 \geq 0 \end{cases}$$

Special solutions

- $u_0 = 0$: $u = 0$
- $u_0 = M > 0$: $\limsup_{t\to+\infty} \sup_{x\in\mathbb{R}^n} u(t,x) \leq 1$
- $u_0 = \delta_0$ and $r = \sup \frac{f(u)}{u}$: for $t \geq 1$,

$$u(t,x) \leq \frac{1}{(4\pi t)^{n/2}} \mathrm{e}^{-\frac{|x|^2}{4dt}+rt} \leq C\mathrm{e}^{-C'(|x|^2-4rdt^2)}$$

Consequences

Nonnegativity, well-posedness, **possible spreading**