Slow diffusion in disordered lattices

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Plan of this talk

1. Phenomenology of disordered lattices:
    - positive temperature
    - zero temperature (spreading of a wave packet)
2. **New results** for the spreading of the wave packet
3. Mathematical results at positive temperature
4. Sub-diffusion (if there is some time left)

Anderson localization in classical chains of oscillators

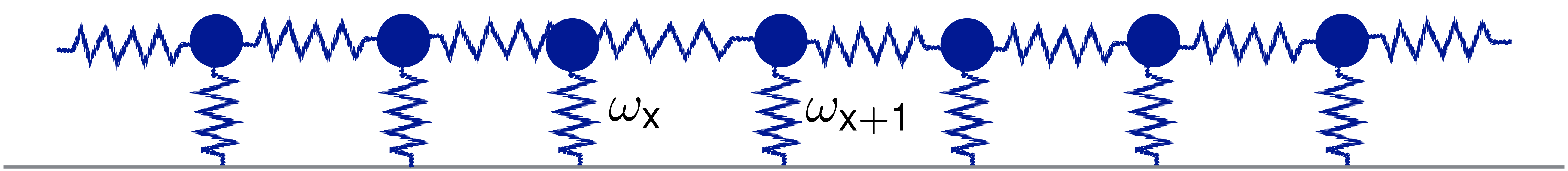
The system

Classical Hamiltonian for *N* coupled oscillators in *1d*:

$$H(q,p) \;=\; \frac{1}{2}\sum_{x=1}^{N}\left(p_x^2+\omega_x^2 q_x^2+g(q_{x+1}-q_x)^2+\lambda q_x^4\right)$$

harmonic — anharmonic

where $(\omega_x)_x$ are i.i.d.



Anderson insulator : *λ=0*

Let us decompose $H = H_0 + \lambda H_1$

$$H_0 = \frac{1}{2}\Big(\langle p, p\rangle + \langle q, (V - g\Delta)q\rangle\Big)$$

with : $V_{x,y} = \delta_{x,y}\omega_x^2$

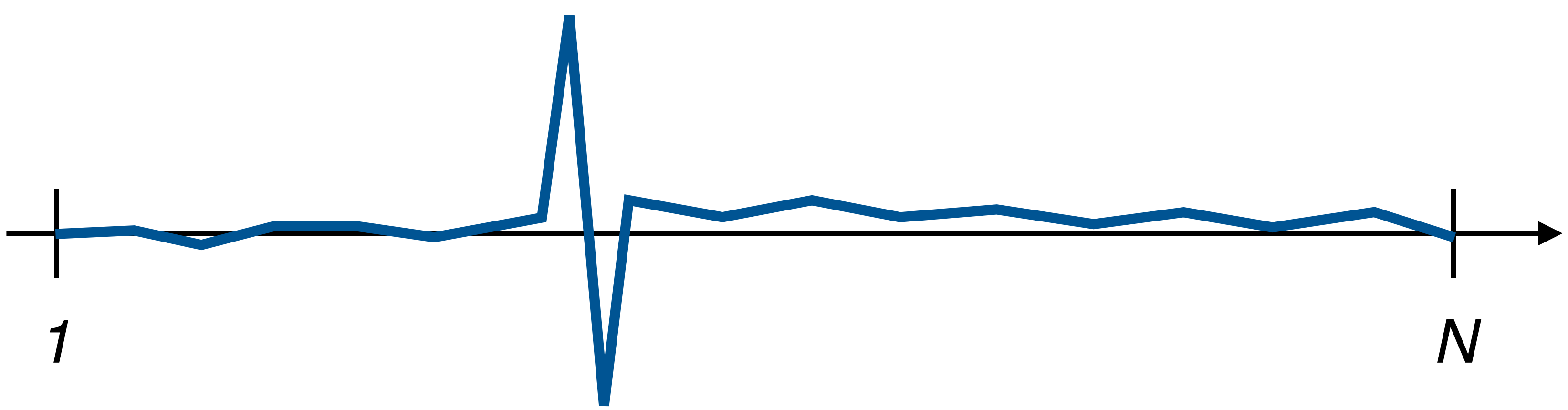
Δ the lattice Laplacian

$\langle\cdot,\cdot\rangle$ the scalar product on $\mathbb{R}^N$

We recognize the Anderson operator *V - gΔ* in *1d*:
all eigenstates are localized !

Localized dynamics

Eigenstates of *V - gΔ* look like this:



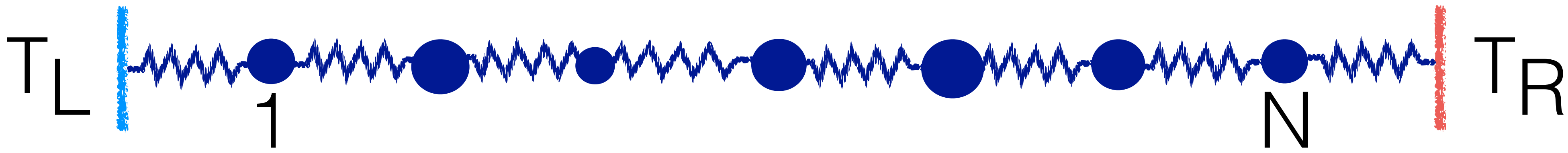
$$\mathsf{E}\left(\sum_E |\langle x, E\rangle \langle E, y\rangle| \right) \leq Ce^{-|x-y|/\xi}$$

Kunz et Souillard ‘80

As a result, the dynamics of the chain is **frozen**.
Let us see this in two set-ups.

1) Positive temperature : No transport

Connecting the system with heat baths



yields an energy flows ***J*** from hot to cold, but

$$J \ \sim \ \Delta T \times e^{-N/\xi}$$

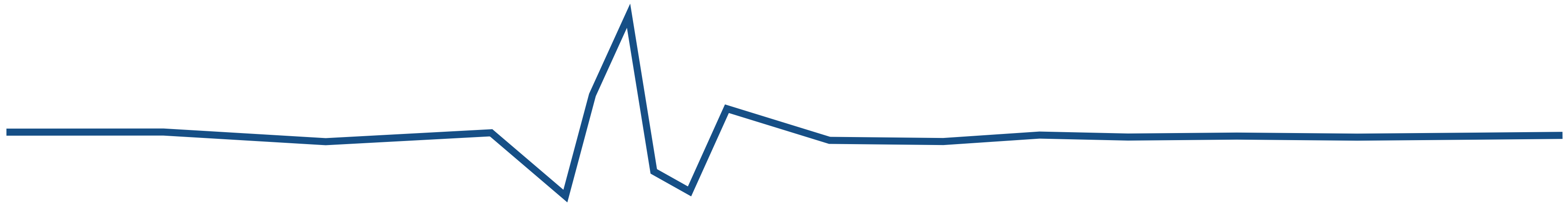
For comparison, in ‘usual’ diffusive systems, we expect

$$J \ \sim \ \frac{\Delta T}{N}$$

2) Zero temperature : No spreading

An initially localized wave packet…

… does never spread over time



What happens when

$\lambda > 0$?

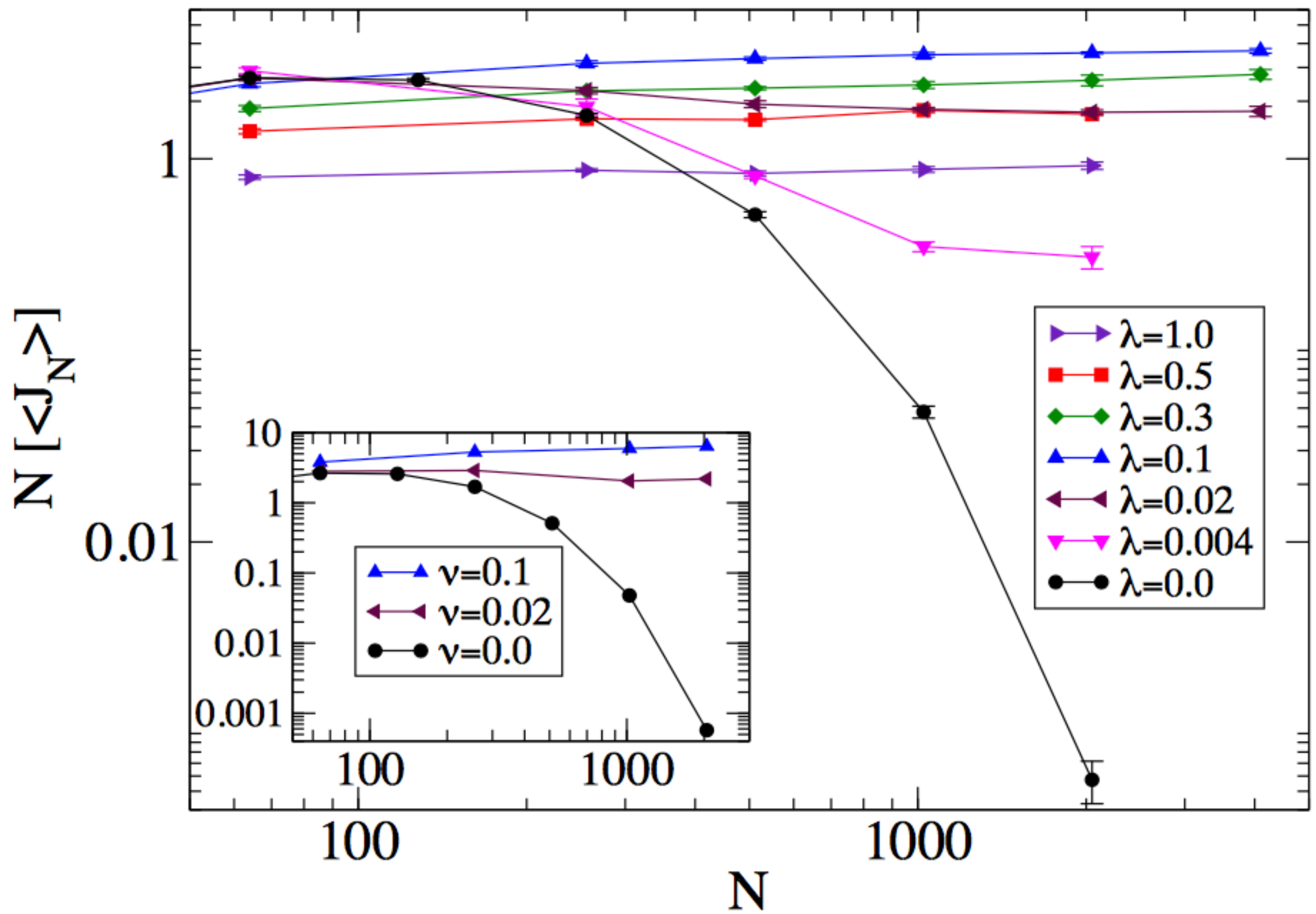
Adding anharmonic interactions : $\lambda>0$

Common expectations:

- **No MBL** for classical systems: anharmonic interactions destroy Anderson localization. Main reason: chaotic spots
- **Very slow** processes involved in the limit $\lambda \to 0$. Transport hard to quantify in this regime
  - Positive temperature: thermal conductivity is **normal**
  - Positive temperature: sub-diffusion in some (very) particular cases
  - Zero temperature: Initially localized wave packet spreads **sub-diffusively**

Normal conductivity…

numerical results for our model:



« Even a small amont of anharmonicity leads to a J~1/N dependence, implying diffusive transport of energy. »

A. Dhar and J.L. Lebowitz, PRL (2008)

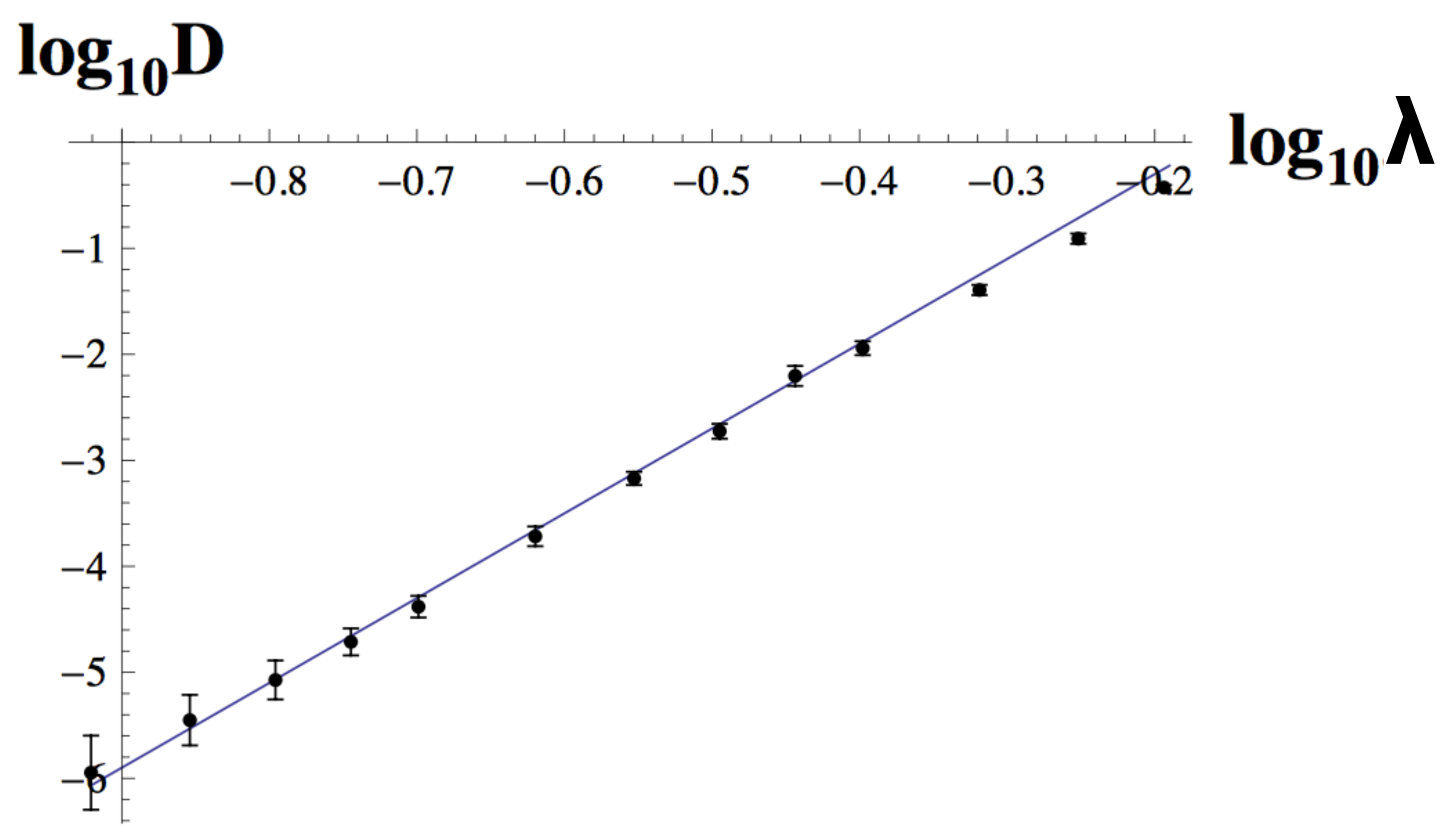
…but very slow rate as $\lambda \to 0$

Numerics in a classical spin chain:

$$H = \sum_x \omega_x \mathbf{S}_x + \lambda \mathbf{S}_x \cdot \mathbf{S}_{x+1}$$

$$J \simeq D(\lambda) \frac{\Delta T}{N}$$

$$D(\lambda) \sim \lambda^8$$



« This suggests that the asymptotic behavior at small interaction λ may be some sort of exponential, rather than power-law behavior. »

V. Oganesyan, **A. Pal** and **D. Huse**, PRB (2009)

in agreement with analytic predictions

Theoretical predictions in the Discrete Non-Linear Schrödinger (DNLS) lattice. Expected to hold for our model as well :

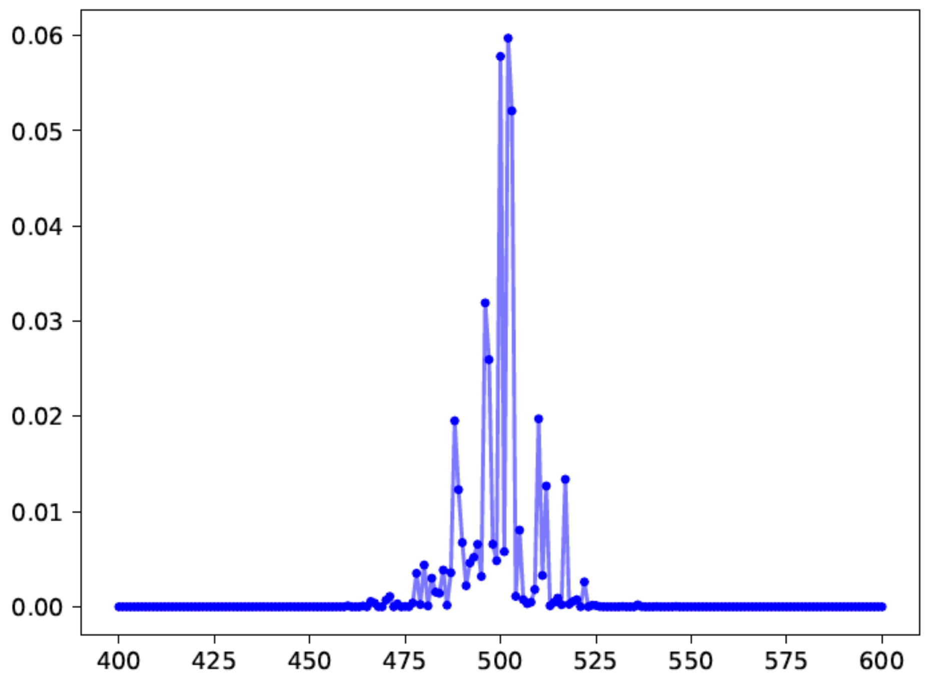
D(λ)→0 faster than any polynomial in ***λ*** as ***λ→0***:

$$\forall n \in \mathbb{N}, \quad D(\lambda) = \mathcal{O}(\lambda^n) \quad \text{as} \quad \lambda \to 0.$$

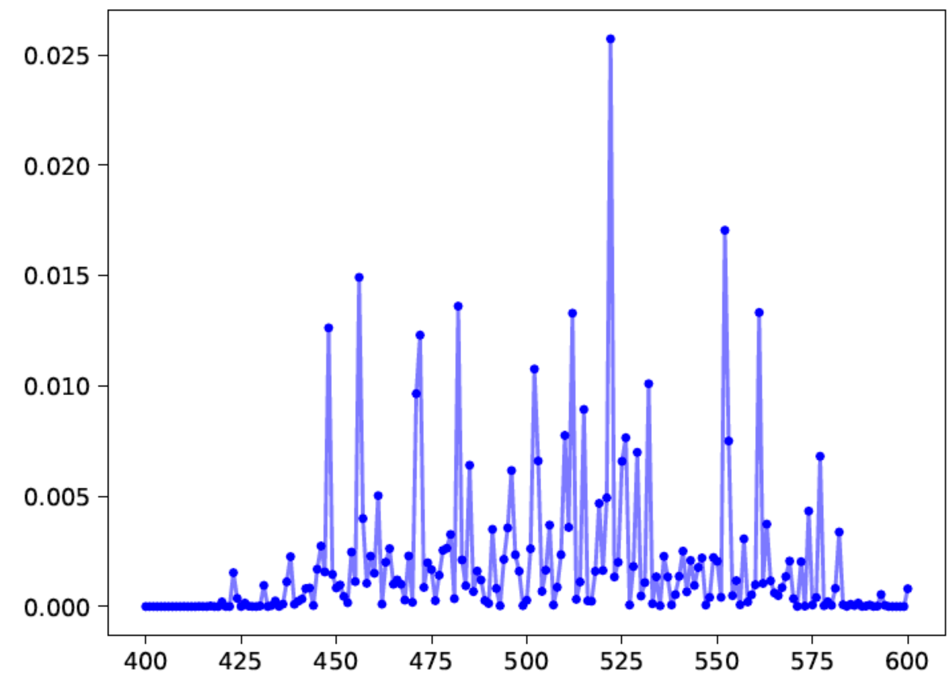
see **D. Basko**, Ann. Phys (2011),
see also **S. Fishman**, **Y. Krivolapov** and **A. Soffer** (2012)

Spreading: analytic predictions

Very slow spreading known rigorously (DNLS):



initial



later ($\tau > 0$)

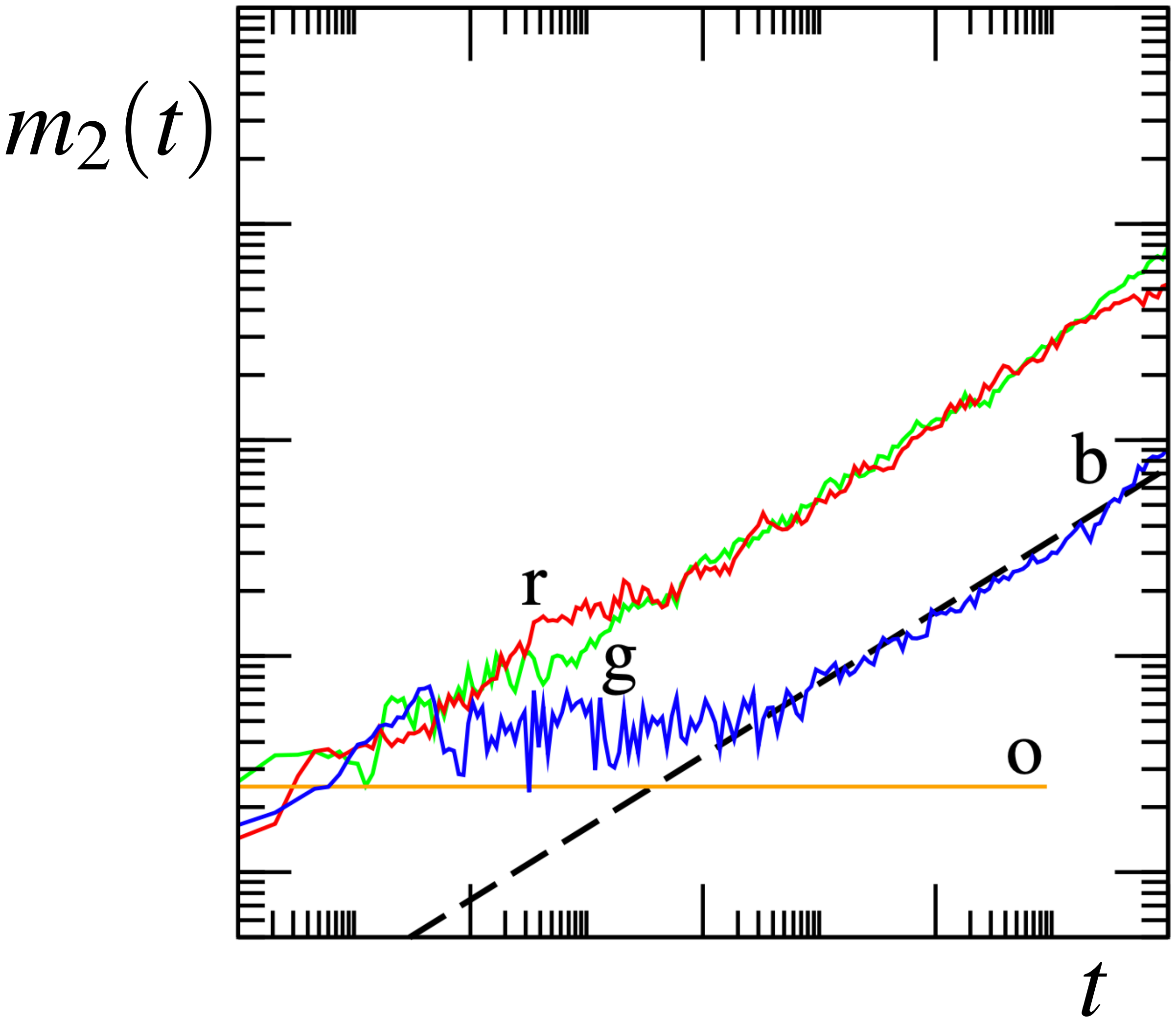
$$\forall n \in \mathbb{N},\ \exists \lambda_0 :\ \lambda < \lambda_0 \quad \Rightarrow \quad \tau(\lambda) \geq \lambda^{-n}$$

This does not settle the long time behavior for fixed λ.

See **W.-M. Wang** and **Z. Zhang** (2009)**, S. Fishman**, **Y. Krivolapov** and **A. Soffer** (2012)
Possible recent improvement by **H. Cong** and **Y. Shi** (preprint 2020)

Spreading: numerics

Numerical results for our model:



$m_2(t)$:

variance of the energy wave packet

b, g, r : from small to large energy

– – – – – – $t^{1/3}$

See **S. Flach** (2014)

Spreading: numerics vs analytic

Numerically, the spreading results in sub-diffusion with exponent 1/3. All rigorous and analytical theories predict that asymptotically the spreading cannot be faster than logarithmic in time. The main difficulty is, that there is no regime of parameters, where analytical and numerical results agree for a long time.

From S. Fishman, Y. Krivolapov and A. Soffer (2012)

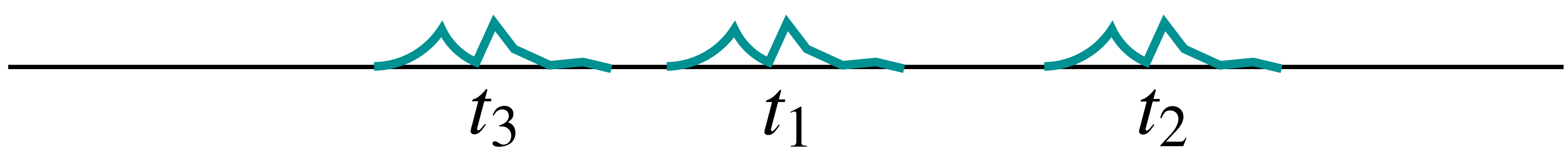
This is the thing I want to fix!

Spreading vs conductivity

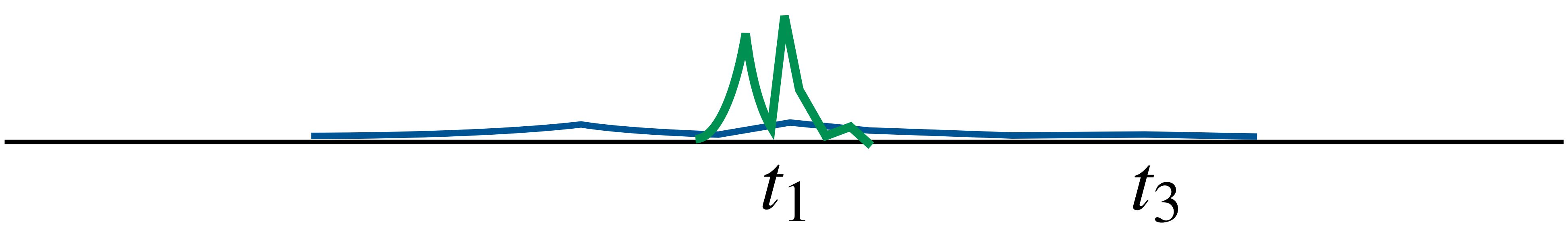
Two scenarios for the ‘spreading’:

1) wandering of a hot spot

not observed



2) proper spreading of the packet



observed, but polynomial spreading clashes with non-polynomial decay of $D(\lambda)$

Why does it clash ?

Assuming some local equilibrium (tested numerically):

$$\partial_t E \;=\; \partial_x \big(D(T,\lambda)\partial_x E \big), \qquad T = T(E)$$

and $\quad T \to 0 \quad$ as $\quad t \to \infty$

In our model: low temperature ~ small λ. Actually

$$D(T,\lambda) \;=\; \overline{D}(T\lambda)$$

and it would hold that

$$m_2(t) \sim t^{1/3} \quad \Rightarrow \quad \overline{D}(T\lambda) \sim (T\lambda)^4 \quad \text{as} \quad T\lambda \to 0.$$

Could we resolve this ?

Preliminary remarks

1. The observed spreading of the wave packet is **very slow**:

$$\sqrt{m_2(t)} \;\sim\; t^{1/6}$$

   This might thus well be just transient.

2. I will move to a different (related) set-up, closer to equilibrium than the spreading experiment (this allows to use some bounds)

Breaking conservation laws

Reminder: model at ***λ=0***:

$$
\begin{aligned}
H_0 &= \frac{1}{2}\big(\langle p, p\rangle + \langle q, (V - g\Delta)q\rangle\big) \\
&= \frac{1}{2}\sum_E \left(|\langle p, E\rangle|^2 + E|\langle q, E\rangle|^2\right) \\
&= \sum_E H_E
\end{aligned}
$$

with

$$
(V - g\Delta)|E\rangle = E\,|E\rangle
$$

The energy of each mode is conserved at ***λ=0***:

$$
\frac{dH_E}{dt} = 0 \qquad \forall E
$$

Breaking conservation laws

Let us consider the full dynamics $H = H_0 + \lambda H_1$ and define

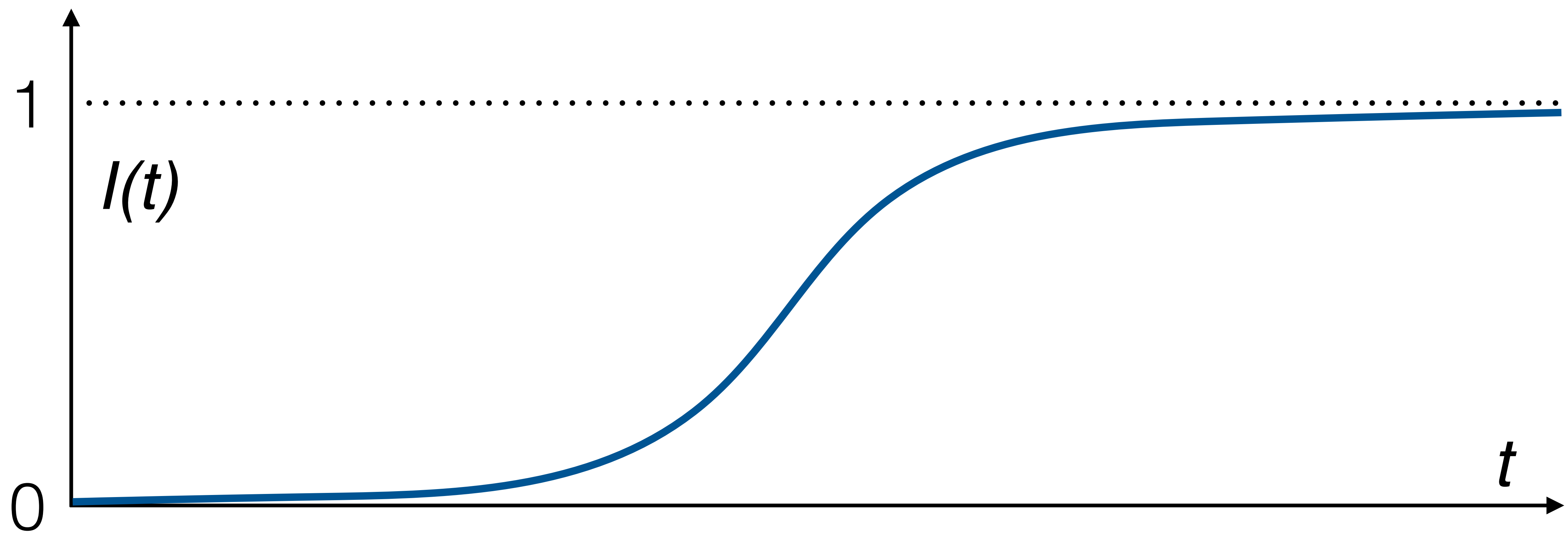
$$I(t) = \frac{1}{N} \sum_E \frac{\langle (H_E(t) - H_E(0))^2 \rangle_T}{2\,\mathrm{var}(H_E)}$$

where $\langle f \rangle_T$ denotes the Gibbs state at temperature ***T***:

$$\langle f \rangle_T = \frac{1}{Z} \int f(q,p) e^{-H(q,p)/T} dq dp$$

(these are invariant states of the dynamics)

Expected behavior of *I(t)*



$$I(0) \;=\; 0 \;\; : \text{by definition}$$

$$I(+\infty) \;=\; 1 \;\; : \text{in the large } N \text{ limit, } \langle H_E(t); H_E(0)\rangle_T \to 0 \text{ as } t \to \infty$$

Scaling relation for *I(t)*

As for the the conductivity ***D***, it holds actually that

$$I(\lambda, T; t) \ = \ \bar{I}(\lambda T; t)$$

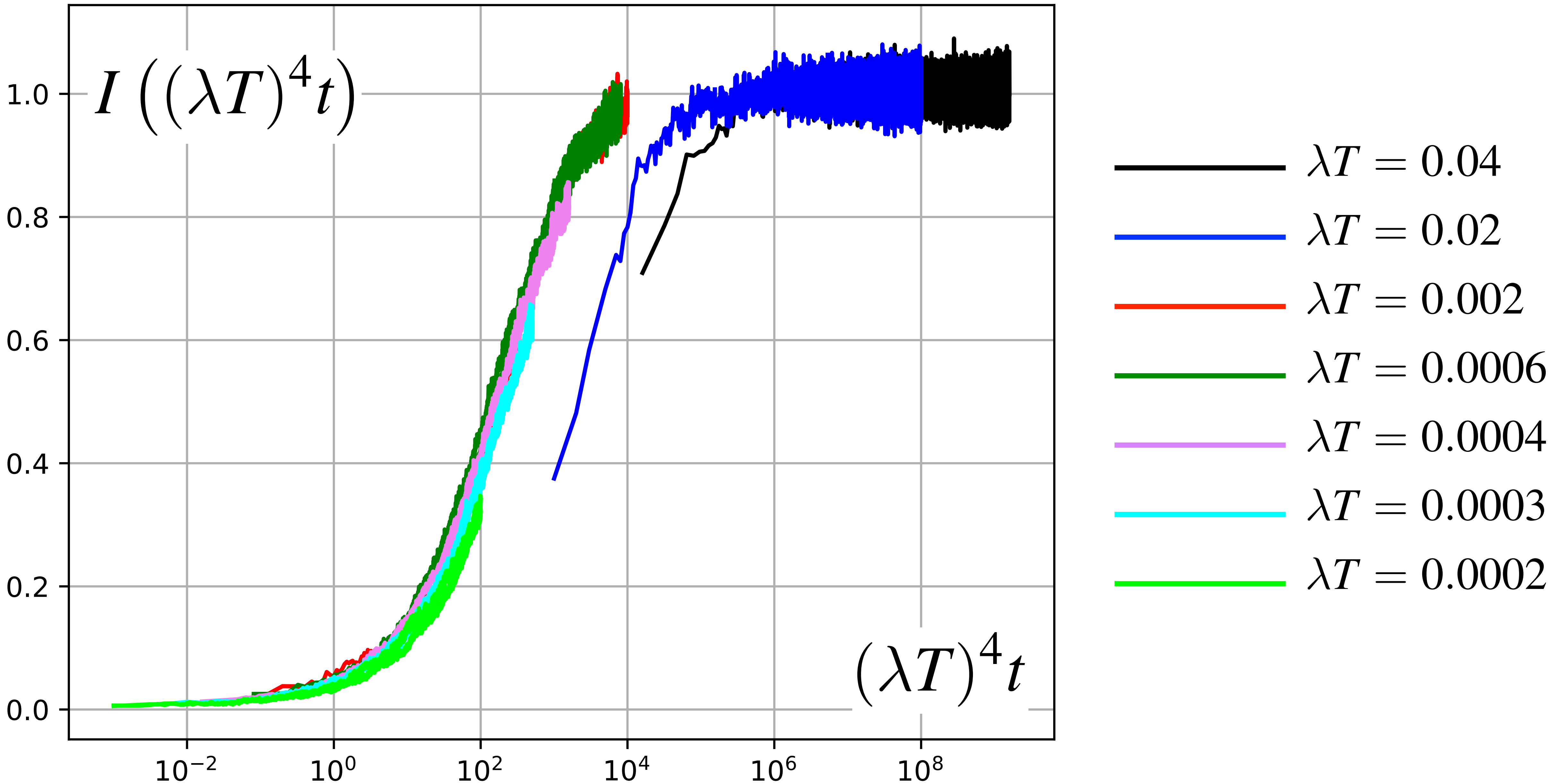
If Flach's results for the spreading are correct, we may expect the scaling relation

$$\bar{I}(\lambda T; t) \ = \ f\left((\lambda T)^4 t\right)$$

We will now

- check via numerics that it holds for some regime in ***λT***,
- map this regime to the time horizon in spreading experiments,
- show that this scaling cannot hold as ***λT→0.***

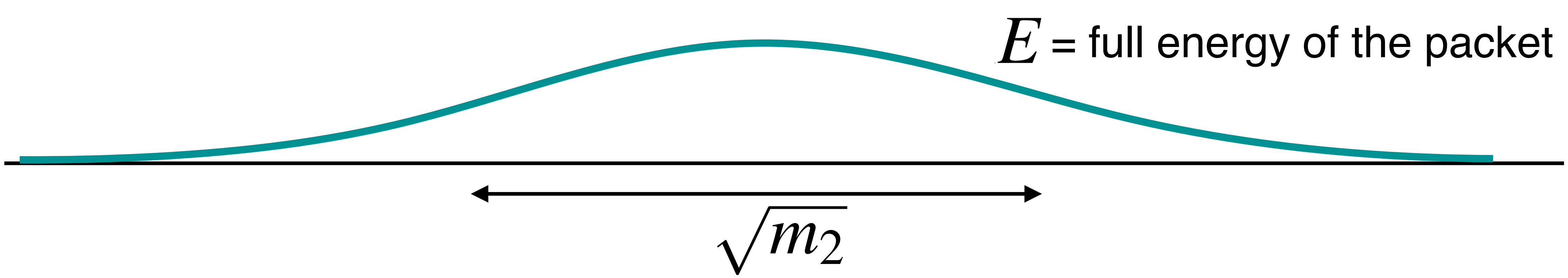
Numercial results



Back to wave packets

Effective temperature of wave packets:

$$\lambda T \longleftrightarrow \frac{\lambda E}{3.3\sqrt{m_2}}$$



Maximal values reached:

λT=0.0005 (Flach), λT=0.0002 (us)

Remark: If $\sqrt{m_2} \sim t^{1/6}$, this requires spreading for $(2/5)^6 \simeq 250$ longer times.

What is the right scaling?

We postulated that

$$I(\lambda T; t) \;=\; f\left((\lambda T)^4 t\right)$$

but we **can show** that

$$\forall n \in \mathbb{N}, \exists C_n \;:\; \qquad I(\lambda T; t) \;\leq\; C_n \left(\lambda^2 + ((\lambda T)^n\, t)^2\right)$$

For the spreading, we infer that $\sqrt{m_2} \sim t^{1/6}$ is incorrect:

Not (yet) a proper theorem

The proof is still incomplete. We need estimates on the spectrum of the **harmonic** system (**λ**=0). E.g.:

$$\forall N > 0, \forall a > 0, \exists b > 0 :$$

$$\mathsf{P}\left(\exists E_1, E_2, E_3, E_4 \in \sigma\left(H_0^{(N)}\right) \; : \; \left|E_1^{1/2} + E_2^{1/2} - E_3^{1/2} - E_4^{1/2}\right| < \frac{1}{N^b}\right) \leq \frac{1}{N^a}$$

where $H_0^{(N)}$ is the harmonic Hamiltonian restricted to a box of size N and where we assume $(E_1, E_2) \neq (E_3, E_4)$ and $(E_2, E_1) \neq (E_3, E_4)$.

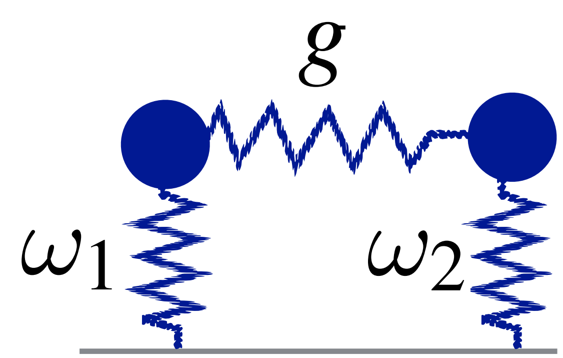
Remark: The needed estimates are not sharp.

Some related results: Minami (1996), Klopp (2010), S. Fishman, Y. Krivolapov and A. Soffer (2008)

Main ingredients of the proof:

1. Frequency mismatch is bad for energy transfer

e.g.: two coupled harmonic oscillators:



the eigenmodes are eigenstates of

$$\begin{pmatrix} \omega_1 & g \\ g & \omega_2 \end{pmatrix}$$

they are *localized* if

$$g \ll |\omega_1 - \omega_2|$$

2. Perturbation theory works as well w or w/o anharmonic interactions at finite order (and this is what we need). Main difference shows up when analyzing the convergence

Some results at positive temperature

Thermal conductivity

Consider (almost) the same system:

$$H = \frac{1}{2}\sum_{x=1}^{N}\left(p_x^2 + \omega_x^2 q_x^2 + \lambda(q_{x+1} - q_x)^2 + \lambda q_x^4\right)$$

Recall that the energy flux is expected to scale as

$$J_N \simeq D\frac{\Delta T}{N}$$

Green-Kubo expression for the conductivity:

$$\begin{aligned} D(\lambda T) &= \lim_{\Delta T\to 0}\lim_{N\to\infty}\frac{NJ_N}{\Delta T} \\ &= \lim_{t\to\infty}\lim_{N\to\infty}\frac{1}{T^2}\left\langle\left(\frac{1}{\sqrt{t}}\int_0^t \frac{1}{\sqrt{N}}\sum_{x=1}^{N-1} j_{x,x+1}(s)\right)^2\right\rangle_T \end{aligned}$$

Difficulty and way out

Problem: We don’t even know whether $D(\lambda T) \ < \ +\infty$

Introduce some **noise** such that

1) Energy still conserved
2) The Gree-Kubo integral is convergent
3) Not too large: Hamiltonian effects dominant

E.g.: add velocity flip

Liouville operator

Noise

$$\mathcal{L} = \mathcal{A} + \gamma \mathcal{S}$$

$$\mathcal{S} f(p,q) \ = \ \sum_x f(\ldots, -p_x, \ldots, q) - f(p,q)$$

Result

Theorem (F.H.):

$$\forall n \in \mathbb{N}, \exists C_n : \quad D(T\lambda, \gamma) \; \leq \; C_n \left(\frac{(T\lambda)^{2n}}{\gamma} + \gamma \right)$$

In particular, with $\gamma = (T\lambda)^n$, we get

$$D(T\lambda, (T\lambda)^n) \; \leq \; C_n (T\lambda)^n$$

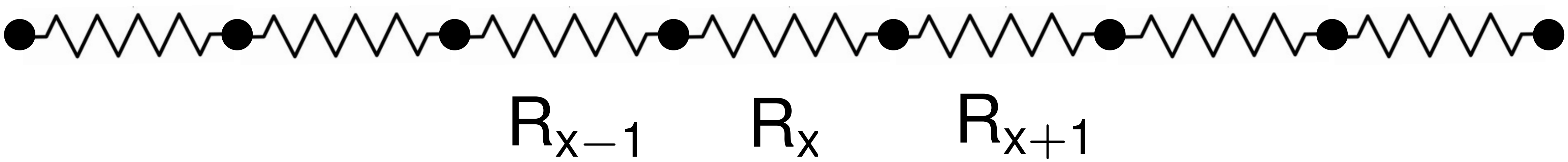
Remark: We **expect** that the noise favors conduction, i.e.

$$D(T\lambda, 0) \; \leq \; C_n (T\lambda)^n$$

Subdiffusion

Phenomenology: Ohm law

Series random resistances (i.i.d.):



Potential difference at the boundary: ΔV

Electrical current:

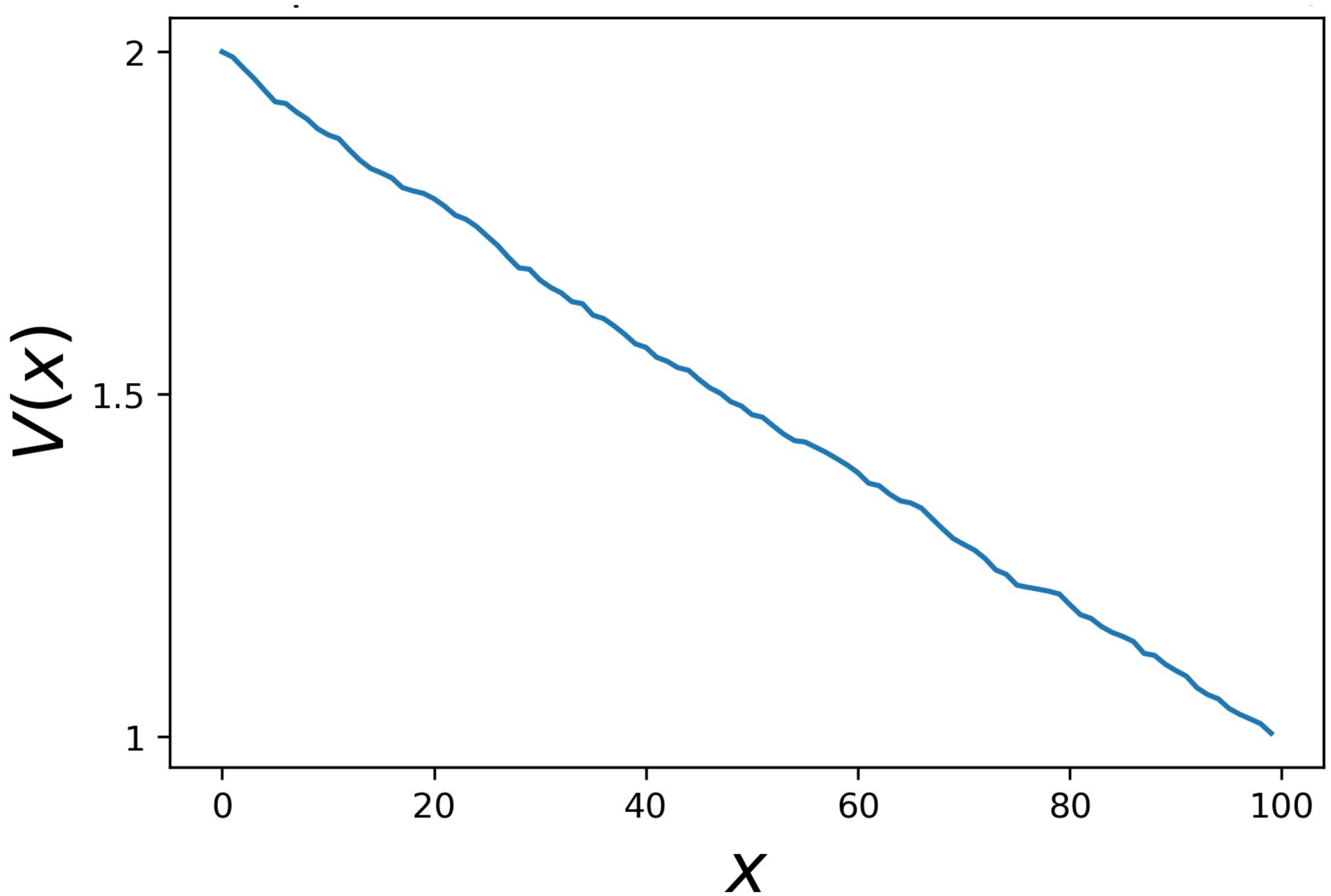
$$J = -\frac{\Delta V}{R} = -\frac{\Delta V}{R_1 + \cdots + R_L}$$

'standard' case

Assume $\overline{R} = E(R_1) < +\infty$

By the law of large numbers: $J \simeq \dfrac{\Delta V}{L\,\overline{R}}$

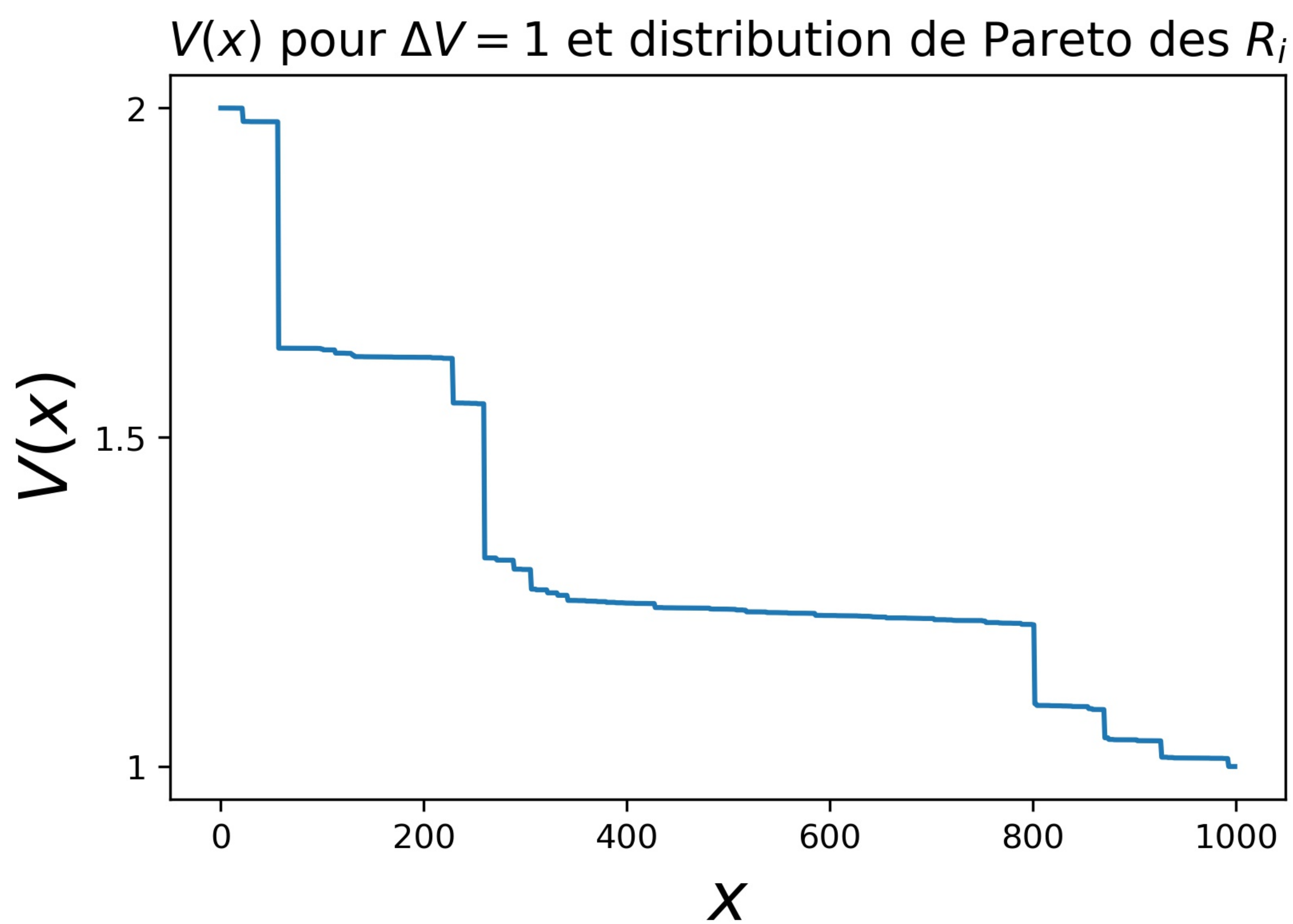
V(x) for ΔV=1 and R_i uniformly distributed



Heavy tails distribution

Assume $\overline{R} = E(R_1) = +\infty$ (ex : Pareto)

$$R_1 + \cdots + R_L \sim \max_i R_i \sim L^a \quad (a > 0)$$



V drops = bottleneck = larges values for R_i

Behavior of the energy current

diffusive : $J \sim L^{-1}$

sub-diffusive: $J \sim L^{-(1+a)}, \quad a > 0$

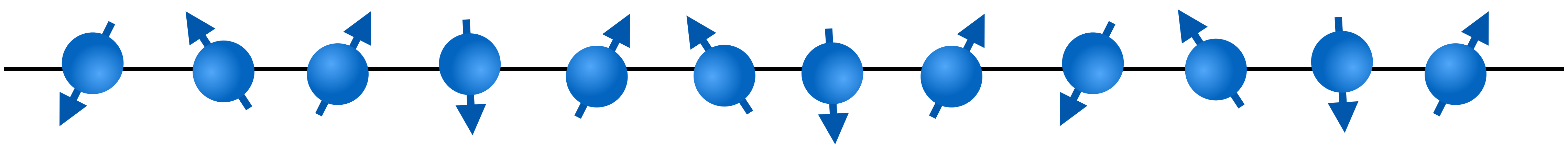
localized : $J \sim e^{-L/\xi}, \quad \xi > 0$

Remark: sub-diffusion rests here on bottlenecks (1d) while localization is a wave phenomenon

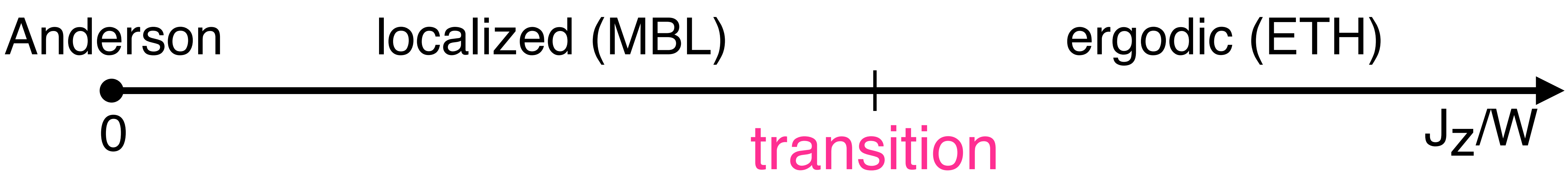
Inspiration: quantum model

Quantum model (ex: XXZ):

$$H = \sum_{i=1}^{L} h_i S_i^z + J_\perp \sum_{i=1}^{L-1} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + J_z \sum_{i=1}^{L-1} S_i^z S_{i+1}^z$$

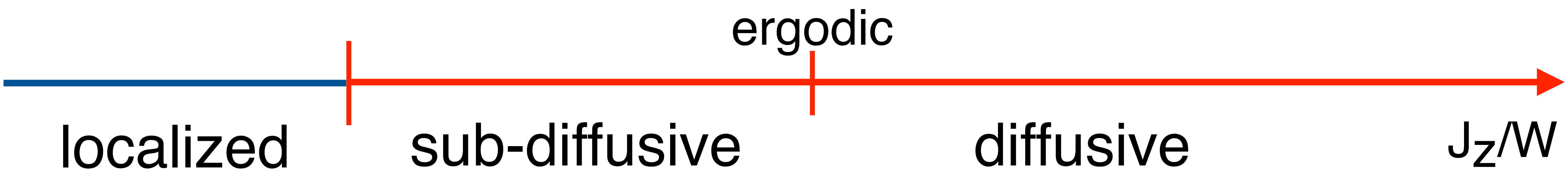


Phase diagram:



Gornyi et al. ’05, Basko et al. ’06, Oganesyan et al. ’07, Serbyn et al. ’13, Huse et aL ’14, Imbrie ’16, etc…

‘transition’ inside the ergodic phase



$J \sim e^{-L/\xi}$ $\quad$ $J \sim L^{-(1+a)}$ $\quad$ $J \sim L^{-1}$

S. Gopalakrishnan et al. ’16, D. Luitz et al. ‘16

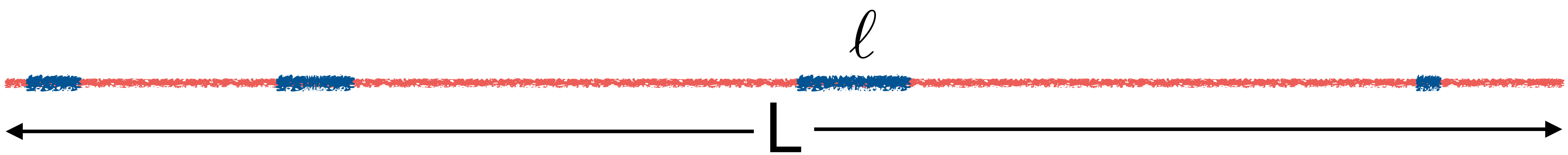
origin of the sub-diffusive phase:
disorder fluctuations (**Griffiths** effects)

blue regions would be localized if they were isolated,
they create vey large resistances

We look for the larges résistance

L : total length

ℓ : length of the largest resistance



$$\ell \sim C \ln L \quad (C \to \infty \text{ as one approaches the transition})$$

$$R(\ell) \sim e^{\ell/\xi}$$

$$J \sim R^{-1} \sim e^{-(C/\xi)\ln L} = L^{-C/\xi}$$

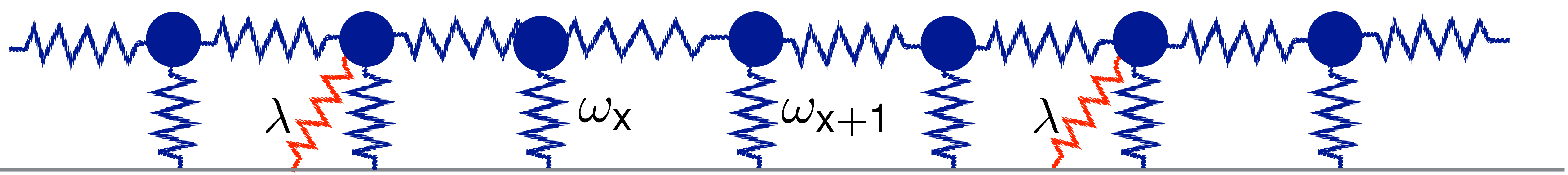
$C/\xi > 1$ near the transition

Classical Hamiltonian with *sparse* interactions

$$H(p,q) \;=\; \frac{1}{2}\sum_{1\leq x\leq L} p_x^2 + \omega_x^2 q_x^2 + g(q_{x+1}-q_x)^2 + \lambda_x q_x^4$$

disordered harmonic chain
=
localized system

anharmonic pinning
=
interaction among modes



λ_x Bernoulli i.i.d. $\qquad P(\lambda_x = 1) = p$

Green-Kubo coefficient

$$D(T) = \frac{1}{T^2} \lim_{t\to\infty} \lim_{L\to\infty} \left\langle \left(\frac{1}{\sqrt{t}} \int_0^t \frac{1}{\sqrt{L}} \sum_{1\leq x\leq L} j_x(s)ds \right)^2 \right\rangle_T$$

$j_x =$ energy current between site x and $x + 1$

$$\langle f\rangle_T = \frac{1}{Z(T)} \int f\, e^{-H/T} dpdq = \text{Gibbs state}$$

Remarks :

In general, we don't know whether $D(T) < +\infty$

Sub-diffusion if $D(T) = 0$

Résult

Theorem (W. De Roeck, S. Olla, F. H.)

Let ξ be the localization length of the harmonic system. If

$$p < 1 - e^{-1/9\xi}$$

then, a.s.

$$D(T) = 0$$

Somewhat more quantitative estimate:

$$D(T) = \lim_{t\to\infty} \frac{C(t)}{t}, \qquad C(t) \le t^{\gamma}, \quad \gamma < 1$$

and one can estimate γ as a function of *p*.

Other models

- Similar results by B. Nachtergaele et J. Reschke (JSP 2021)
- Analogous theorem for quantum chains (fermions or spins):

$$H = \sum_{1 \leq x \leq L} \omega_x n_x + J(c_x^\dagger c_{x+1} + c_x c_{x+1}^\dagger) + \lambda_x n_x n_{x+1}$$

- Ongoing research to extend this result for quantum chains with weak but non-sparse interactions

- The claim is expected to be wrong for classical systems and weak (non-sparse) interactions:

cf. A. Dhar et J. Lebowitz, PRL 2004