On recent developments on the long-term dynamics for the equivariant self-dual Chern–Simons–Schrödinger equation

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Chern–Simons–Schrödinger equation

The Chern–Simons–Schrödinger equation is

- introduced by the physicists **Jackiw** and **Pi** '90, as a nonrelativistic Chern–Simons gauge field theory, which admits the **self-dual** structure when the coupling strength is critical.
- $U(1)$ gauge-covariant 2D cubic NLS ($i\partial_t\psi + \Delta\psi + |\psi|^2\psi = 0$), so it is similar to 2D cubic NLS. (symmetries/conservation laws, ...)
- The Chern–Simons action has been employed, for example, in condensed matter physics to describe some planar physics, e.g., quantum Hall effect and high temperature superconductivity.

In the **self-dual** case, under equivariance symmetry, CSS turns out to be an interesting mathematical model:

- **Strong rigidity in asymptotic behavior of solutions**:
  Soliton resolution, non-existence of bubble trees,
- **Blow-up dynamics and rotational instability**:
  Completely different near-soliton dynamics compared to that of NLS!
  Rather similar to **wave maps** or **Schrödinger maps** from $\mathbb{R}^{1+2}$ into $\mathbb{S}^2$.

Chern–Simons–Schrödinger equation

- Unknowns:
  - matter field $\phi : \mathbb{R}^{1+2} \to \mathbb{C}$
  - gauge fields $A_\alpha : \mathbb{R}^{1+2} \to \mathbb{R}$, $\alpha \in \{t, 1, 2\}$.

  Let $\mathrm{D}_\alpha := \partial_\alpha + iA_\alpha$ and $F_{\alpha\beta} := \partial_\alpha A_\beta - \partial_\beta A_\alpha$.
- Equation:

$$\begin{cases} i\mathrm{D}_t\phi + \mathrm{D}_j\mathrm{D}_j\phi + g|\phi|^2\phi = 0, \\ F_{t1} = -\mathrm{Im}(\overline{\phi}\mathrm{D}_2\phi), \\ F_{t2} = \mathrm{Im}(\overline{\phi}\mathrm{D}_1\phi), \\ F_{12} = -\frac{1}{2}|\phi|^2, \end{cases}$$

- **Self-dual** case: $g = 1$.
- $\exists L^2$-scaling symmetry, mass/energy conservation, pseudoconformal symmetry, virial identities, ...

Coulomb gauge

Gauge invariance:

$$(\phi, A) \quad \to \quad (e^{i\chi}\phi, A - d\chi).$$

We will fix the **Coulomb gauge condition**:

$$\partial_1 A_1 + \partial_2 A_2 = 0$$

- A_α is determined by ϕ, i.e., $A_\alpha = A_\alpha[\phi]$
- **Hamiltonian structure**: CSS reduces to a single equation of ϕ

$$\partial_t \phi = -i\nabla E[\phi], \qquad E[\phi] := \frac{1}{2}\int |\mathrm{D}_x \phi|^2 - \frac{g}{4}\int |\phi|^4.$$

- **LWP**: H^1 in Coulomb gauge (Lim '18), small data H^{0+} in the heat guage (Liu-Smith-Tataru '14). L^2-critical well-posedness is still open.
- Other known results: existence of solitons and explicit blow-up solutions, formal derivation of log-log blow-up for $g > 1$, small data decay and scattering, etc.

Symmetry reduction: equivariance

Equivariance ansatz: (r, θ) polar coordinates

$$\phi(t, x) = e^{im\theta} u(t, r) \quad (m \in \mathbb{Z}\text{: equivariance index})$$

The main equation: (equation depends on m)

$$i\partial_t u + \left(\partial_{rr} + \frac{1}{r}\partial_r\right) u - \left(\frac{m + A_\theta}{r}\right)^2 u - A_t u + g|u|^2 u = 0$$

with connection components

$$A_\theta = -\frac{1}{2}\int_0^r |u|^2 r' dr' \qquad \text{and} \qquad A_t = -\int_r^\infty (m + A_\theta)|u|^2 \frac{dr'}{r'}.$$

- **Non-local** nonlinearity (both from A_θ and A_t)
  $A_\theta(0) = 0$ but $A_\theta(\infty) = -\frac{1}{2}\int_0^\infty |u|^2 r dr < 0$.
- No derivative nonlinearity (L^2-critical well-posedness via Strichartz estimates)

Symmetries and conservation laws

CSS has symmetries and conservation laws analogous to the L^2-critical NLS:

- Conservation of Energy and Charge:

$$E[u] = \int \frac{1}{2}|\partial_r u|^2 + \frac{1}{2}\Big(\frac{m + A_\theta}{r}\Big)^2 |u|^2 - \frac{g}{4}|u|^4 \qquad M[u] = \int \frac{1}{2}|u|^2$$

- L^2-scaling symmetry:

$$\frac{1}{\lambda} u\Big(\frac{t}{\lambda^2}, \frac{r}{\lambda}\Big)$$

- Pseudoconformal symmetry: (reverts time $t \mapsto -\frac{1}{t}$, scale $r \mapsto \frac{r}{t}$)

$$\frac{1}{t} e^{i\frac{t}{4}|\frac{r}{t}|^2} u\Big(-\frac{1}{t}, \frac{r}{t}\Big)$$

- Virial identities also hold.

Self-duality and static solution Q

From now on, $g = 1$ (self-dual case).

- **Self-dual expression** of energy: ↓ called Bogomol'nyi operator

$$E[u] = \frac{1}{2} \int |\mathrm{D}_u u|^2, \qquad \mathrm{D}_u u := \Big(\partial_r - \frac{m + A_\theta[u]}{r}\Big) u.$$

- **Static solution** (i.e., time-independent solution) can be found by solving the **Bogomol'nyi equation** $\mathrm{D}_Q Q = 0$:

$$Q(r) = \sqrt{8}(m+1) \frac{r^m}{1 + r^{2(m+1)}} \quad \text{for each } m \geq 0.$$

(unique up to phase rotation/L^2-scaling)

- The self-dual structure still exists without symmetry, and the Bogomol'nyi equation is connected to Liouville's equation. (Jackiw–Pi '90)

Brief comparison with NLS

(CSS)	(NLS)
$E[u] = \frac{1}{2}\int \lvert \mathrm{D}_u u\rvert^2 \geq 0$	$E[\psi] = \frac{1}{2}\int \lvert\nabla\psi\rvert^2 - \frac{1}{4}\int\lvert\psi\rvert^4$
$u(t,r) = Q(r)$	$\psi(t,x) = e^{it}R(r)$
$\mathrm{D}_Q Q = 0$	$-\Delta R + R - \lvert R\rvert^2 R = 0$
$\nabla E[Q] = 0$	$\nabla E[R] + \nabla M[R] = 0$
Polynomial decay $r^{-(m+2)}$	Exponential decay

Linearized operators for (CSS) and (NLS) are completely different. Generalized null space relations:

(CSS)

$$i\mathcal{L}_Q \rho = iQ \quad i\mathcal{L}_Q i\frac{r^2}{4}Q = \Lambda Q,$$
$$i\mathcal{L}_Q iQ = 0 \quad i\mathcal{L}_Q \Lambda Q = 0.$$

(NLS)

$$i\mathcal{L}_{\mathrm{NLS}}\rho_{\mathrm{NLS}} = ir^2 R,$$
$$i\mathcal{L}_{\mathrm{NLS}} i\frac{r^2}{4}R = \Lambda R,$$
$$i\mathcal{L}_{\mathrm{NLS}}\Lambda R = -2iR,$$
$$i\mathcal{L}_{\mathrm{NLS}} iR = 0$$

Dynamics below the threshold

The ground state Q plays a pivotal role in large data dynamics.

- **Subthreshold dynamics**
  If $m \geq 0$, $M[u] < M[Q]$, and L^2-data (Liu–Smith '16),
  $\Rightarrow$ GWP/Scattering (c.f. Kenig–Merle '06 and Killip–Tao–Visan '09)
- **Threshold dynamics**
  If $M[u] = M[Q]$ and H^1-data (Liu–Li '20), then u is either (up to symmetries)

$$\begin{cases} \text{(i) global static solution } Q(r), \\ \text{(ii) explicit blow-up solution } S(t,r) = \frac{1}{|t|} Q\left(\frac{r}{|t|}\right) e^{-i\frac{r^2}{4|t|}}, \text{ or} \\ \text{(iii) global-in-time and scatters.} \end{cases}$$

(c.f. Merle '93, Killip–Li–Visan–Zhang '09)

The story so far is very similar to the case of NLS.

Soliton resolution

- General large data dynamics? We would like to believe *the soliton resolution conjecture*: (under generic assumptions on data)

  Asymptotically in time, any solutions decompose into
  the sum of modulated solitons and a radiation.

  (Modulated means L^2-scaled and phase-rotated)

- Soliton resolution has been known for some completely integrable models.
- For non-integrable models? Very recently, soliton resolution is proved for radial critical NLW and equivariant wave maps (Duyckaerts, Kenig, Merle, Martel, Collot, and Jendrej, Lawrie).

(CSS) is in fact very special! Soliton resolution for (CSS) can be proved (in a weighted Sobolev class):

- Consequence of the **self-duality** and **non-local** nonlinearities,
- The dynamics is very rigid: **no multi-soliton**.

Let us recall and fix the notation.

- $m \in \mathbb{Z}$ is the equivariance index, $\phi(t,r) = e^{im\theta}u(t,r)$, (CSS) depends on m.
- $E[u]$, $M[u]$: energy and charge (or, L^2-mass) of u,
- $\mathcal{C}$: pseudoconformal transform, reverts time $t \mapsto -\frac{1}{t}$, scales $r \mapsto \frac{r}{|t|}$.
- $Q = Q(r)$: static solution, not exist when $m < 0$, spatial decay $r^{-(m+2)}$ when $m \geq 0$.
- $S(t) = [\mathcal{C}Q](t)$: the pseudoconformal transform of Q, finite-time blow-up solution.
- $H^{1,1} = H^1 \cap r^{-1}L^2$: weighted Sobolev space
- H^1_m, $H^{1,1}_m$: spaces restricted to m-equivariant functions.
- $\Delta^{(m)} = \partial_{rr} + \frac{1}{r}\partial_r - \frac{m^2}{r^2}$: Laplacian acting on m-equivariant functions.

Soliton resolution: H^1 finite-time blow-up

Theorem (K.–Kwon–Oh, H^1 finite-time blow-up solutions)

(Case $m < 0$) there is no finite-time blow-up H^1-solution.
(Case $m \geq 0$) If u is a H^1_m-solution to (CSS) that blows up in finite time T, then there exist scale parameter $\lambda(t)$, phase parameter $\gamma(t)$, and an asymptotic profile $z^*(r) \in L^2$ satisfying

$$u(t,r) - \frac{e^{i\gamma(t)}}{\lambda(t)} Q\Big(\frac{r}{\lambda(t)}\Big) \to z^*(r) \qquad \text{in } L^2$$

as $t \to T$ and

$$\text{(further regularity)} \qquad z^* \in H^1,$$

$$\text{(bound for the blow-up rate)} \qquad \lambda(t) \lesssim_{M[u]} \begin{cases} \frac{\sqrt{E[u]}(T-t)}{|\log(T-t)|^{\frac{1}{2}}} & \text{if } m = 0, \\ \sqrt{E[u]}(T-t) & \text{if } m \geq 1. \end{cases}$$

Soliton resolution: global $H^{1,1}$ solutions

Theorem (K.–Kwon–Oh, $H^{1,1}$ global solutions)

(Case $m < 0$) Any $H^{1,1}_m$-solution scatters.
(Case $m \geq 0$) If u is a forward-in-time global $H^{1,1}_m$-solution to (CSS), then either u scatters forward-in-time or there exist scale parameter $\lambda(t)$, phase parameter $\gamma(t)$, and a scattering profile $u^*(r) \in L^2$ satisfying

$$u(t,r) - \frac{e^{i\gamma(t)}}{\lambda(t)} Q\Big(\frac{r}{\lambda(t)}\Big) - e^{it\Delta^{(-m-2)}} u^* \to 0 \qquad \text{in } L^2$$

as $t \to +\infty$ and

$$\text{(further regularity)} \qquad u^* \in H^{1,1},$$

$$\text{(bound for the blow-up rate)} \qquad \lambda(t) \lesssim_{M[u]} \begin{cases} \frac{\sqrt{E[\mathcal{C}u]}}{|\log t|^{\frac{1}{2}}} & \text{if } m = 0, \\ \sqrt{E[\mathcal{C}u]} & \text{if } m \geq 1. \end{cases}$$

Comments on the result

- (*Dynamics of $m < 0$*) We show that the equation is **defocusing** in the sense that

$$E[u] \sim_{M[u]} \|u\|_{\dot{H}^1_m}^2.$$

  In particular, there is no (H^1) solitons when $m < 0$.

- (*Nonexistence of multi-solitons*) At most one soliton can appear! Indeed, as a consequence of the **self-duality** and **non-locality**, we observe a **defocusing** nature, i.e., the strict positivity of the energy, **in the exterior** of the soliton profile.

  These features are very special, and are not expected for 2D cubic NLS.

- (*Regularity assumptions on data*) Using the pseudoconformal transform, the global case can always be reduced to the finite-time blow-up case, if one assumes $H^{1,1}$ for the global case.

- (*Bounds for scaling* $\lambda(t)$) When $m \geq 1$, the bounds ($\lambda(t) \lesssim T-t$ or $\lambda(t) \lesssim 1$) are saturated by $S(t)$ and Q. An interesting open question is the construction or the non-existence of the blow-up rates not saturating these bounds.

  When $m = 0$, the improved bounds ($\lambda(t) \lesssim |\log(T-t)|^{-\frac{1}{2}}(T-t)$ or $\lambda(t) \lesssim |\log t|^{-\frac{1}{2}}$) do not include $S(t)$ and Q. This is not a contradiction because $S(t) \notin H^1$ and $Q \notin H^{1,1}$ when $m = 0$. Moreover, there exist finite-time blow-up solutions with the scenarios (K.–Kwon–Oh)

$$\lambda(t) \sim \frac{T-t}{|\log(T-t)|^2} \qquad \text{and} \qquad \lambda(t) \sim \frac{(T-t)^p}{|\log(T-t)|} \text{ for all } p > 1.$$

- (*Comparison with mass-critical NLS*) For the mass-critical NLS, such a result is available in the regime

$$M[R] < M[\psi] < M[R] + \alpha^*, \qquad 0 < \alpha^* \ll 1,$$

  by the works of Merle and Raphaël. For CSS, we could prove a similar result **without mass restriction** on solutions.

  There is no log-log blow-up for (CSS), though such log-log blow-up is expected for the focusing non-self-dual CSS ($g > 1$).

Blow-up dynamics

From now on, we would like to study **refined descriptions** of the dynamics. We focus on the **finite-time blow-up** for **finite energy** (i.e., H^1) solutions:

$$u(t,r) - \frac{e^{i\gamma(t)}}{\lambda(t)} Q\Big(\frac{r}{\lambda(t)}\Big) \to z^*(r) \quad \text{as} \quad t \to 0.$$

Recall:

- Necessarily $m \geq 0$
- $z^* \in H^1$ ($z^*(r)$ called asymptotic profile)
- $\lambda(t) \lesssim \begin{cases} |\log(T-t)|^{-\frac{1}{2}}(T-t) & \text{if } m=0, \\ T-t & \text{if } m \geq 1. \end{cases}$
- $Q(r)$: static solution with spatial decay $r^{-(m+2)}$
- $S(t,r) = [\mathcal{C}Q](t,r) = \frac{1}{|t|} Q(\frac{r}{|t|}) e^{-i\frac{r^2}{4|t|}}$ has finite energy if and only if $m \geq 1$.

Pseudoconformal blow-up solutions, $m \geq 1$

When $m \geq 1$, there is an explicit finite-time blow-up solution $S(t)$, which satisfies

$$S(t,r) - \frac{1}{|t|} Q\Big(\frac{r}{|t|}\Big) \to 0 \quad \text{in } L^2 \quad \text{as } t \to 0.$$

We study the **pseudoconformal blow-up solutions**

$$u(t,r) - \frac{e^{i\gamma(t)}}{\lambda(t)} Q\Big(\frac{r}{\lambda(t)}\Big) \to z^*(r) \quad \text{in } L^2 \quad \text{as } t \to T^-,$$

where $\lambda(t) \approx C(u) \cdot (T-t)$ (i.e., linear), and the dynamics nearby. First,

Theorem (K.–Kwon)

Let $m \geq 1$*. Given the asymptotic profile* $z^*(r)$ *that is small, smooth, and degenerate at the origin (i.e.,* $|z^*(r)| \lesssim r^K$*), there exists a solution* $u(t,r)$ *such that*

$$u(t,r) - \frac{1}{|t|} Q\Big(\frac{r}{|t|}\Big) \to z^*(r) \quad \textit{as} \quad t \to 0.$$

Remark:

- (CSS)-analogue of **Bourgain–Wang** solutions in (NLS).

Instability of pseudoconformal blow-up

Moreover, this pseudoconformal blow-up is unstable:

Theorem (K.–Kwon)

With the same hypothesis as before, there exists a continuous one-parameter family of solutions $\{u^{(\eta)}\}_{\eta\geq 0}$ *such that*

$$u^{(\eta)}(t,r) \approx \frac{e^{i\gamma^{(\eta)}(t)}}{\lambda^{(\eta)}(t)} Q\Big(\frac{r}{\lambda^{(\eta)}(t)}\Big) + z^*(r) \text{ for } t \text{ near } 0,$$

where $\lambda^{(\eta)}$ *and* $\gamma^{(\eta)}$ *are given in the next slide, and* $\{u^{(\eta)}\}_{\eta\geq 0}$ *satisfies*

- $u^{(0)}$ *is the pseudoconformal blow-up solution in the previous slide,*
- *If* $\eta \neq 0$, $u^{(\eta)}$ *scatters both forwards and backwards in time.*

Remark: The theorem is only proved for $\eta \geq 0$, but it should also hold for $\eta \leq 0$.

Rotational instability

The most interesting part is its instability mechanism: **rotational instability!**

$$\lambda^{(\eta)}(t) = (t^2 + \eta^2)^{\frac{1}{2}},$$
$$\gamma^{(\eta)}(t) = \mathrm{sgn}(\eta)(m+1)\left\{ \tan^{-1}\left(\frac{t}{|\eta|}\right) - \frac{\pi}{2} \right\},$$

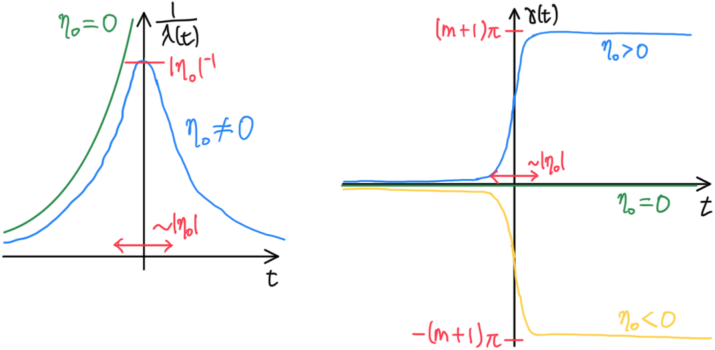

Figure: **Graphs of $\frac{1}{\lambda(t)}$ and $\gamma(t)$**

Rotational instability

which is completely different from the NLS case:

Theorem (Merle–Raphaël–Szeftel '13)

There exists a continuous one-parameter family of radial solutions $\{u^{(\eta)}\}$ *to (NLS) satisfying*

$$u^{(\eta)}(t,r) \approx \frac{e^{i\gamma^{(\eta)}(t)}}{\lambda^{(\eta)}(t)} R\Big(\frac{r}{\lambda^{(\eta)}(t)}\Big) + z^{*}(r) \text{ for } t \text{ near } 0,$$

and,

- $u^{(0)} = u$ *is the Bourgain–Wang (or, pseudoconformal blow-up) solution,*
- *For* $\eta > 0$, $u^{(\eta)}$ *scatters both forwards and backwards in time,*
- *For* $\eta < 0$, $u^{(\eta)}$ *scatters backwards in time but* ***blows up*** *forwards in time under the* ***log-log regime****.*

Rotational instability is not a unique feature of (CSS)!

- formally observed in other critical geometric equations (e.g. **van den Berg–Williams** '13, **Merle–Raphaël–Rodnianski** '13.)

Codimension-one property of pseudoconformal blow-up, $m \geq 1$

We complement our instability result by showing that, the pseudoconformal blow-up can occur from a codimension-one set of initial data:

Theorem (K.–Kwon)

Let $m \geq 1$*. There is a codimension one set of (smooth) initial data* $u_0(r)$ *such that the forward-in-time evolution* $u(t, r)$ *is a pseudoconformal blow-up solution.*

Rotational instability conjecture: There is a codimension-1 manifold $\mathcal{M}$ of initial data such that

- $u_0 \in \mathcal{M} \to$ pseudoconformal blow-up
- $u_0 \notin \mathcal{M}$ but near to $\mathcal{M} \to$ global, scattering, exhibits rotational instability.

Linear conjugation identity: We found a simple but remarkable algebraic identity that formally connects (CSS) to (WM) or (SM) at the linear level. This allows us to find a hidden **repulsivity** property in (CSS).

Smooth finite-energy blow-up solutions, $m = 0$

The radial case ($m = 0$) turns out to be the most delicate.
- $m = 0$ most physically relevant, Q being the ground state without symmetry,
- $S(t) \notin H^1$ and the pseudoconformal blow-up is **ruled out** for **finite energy** solutions.

Thus it is natural to ask if finite energy finite-time blow-up solutions exist.

Theorem (K.–Kwon–Oh)

Let $m = 0$*. There is a codimension one set of smooth finite energy radial initial data* $u_0(r)$ *such that the forward-in-time evolution* $u(t, r)$ *blows up in finite time (say* $T < \infty$*) and*

$$u(t,r) - \frac{e^{i\gamma^*}}{\lambda(t)} Q\Big(\frac{r}{\lambda(t)}\Big) \to z^* \text{ in } L^2, \quad \lambda(t) \approx \lambda^* \frac{T-t}{|\log(T-t)|^2}$$

as $t \to T$*, for some* $\gamma^* \in \mathbb{R}$*,* $\lambda^* \in \mathbb{R}_+$*, and* $z^* \in H^1$*.*

Remarks:

- One can take $u_0 \in C_c^\infty$ and $\|u_0 - Q\|_{L^2} \ll 1$.
- Infinite-time blow-up $\lambda(t) \sim (\log t)^2$ by the pseudoconformal transform.

Comments on the result

Finite energy solution

- $S(t)$ has infinite energy
  $\Rightarrow$ The (conjugated) linearized operator has a **zero resonance**
  $\Rightarrow$ a **logarithmic correction** to the blow-up rate.
  (c.f. **Raphaël–Rodnianski** '12, **Merle–Raphaël–Rodnianski** '13)
- For (NLS), the exact self-similar profile barely fails to lie in L^2
  $\Rightarrow$ log-log correction to the self-similar rate (**Merle-Raphaël**)

Comparison with WM or SM

- Q for (CSS) is similar to the harmonic maps in (WM) or (SM).
- Exactly the same blow-up rate as 1-equivariant (SM) of [**MRR**13].
- Linear conjugation identity of [2].

Continuum of blow-up rates, $m = 0$

Finally, when $m = 0$, there is a continuum of possible blow-up rates.

Theorem (K.–Kwon–Oh)

Let $m = 0$*. For* $q \in \mathbb{C} \setminus \{0\}$ *and* $\mathrm{Re}(\nu) > 0$*, set*

$$z^*(r) = qr^{\nu}\chi(r).$$

Then, there exists a finite energy blow-up solution $u(t, r)$ *such that*

$$u(t,r) - \frac{e^{i\gamma_{q,\nu}(t)}}{\lambda_{q,\nu}(t)} Q\Big(\frac{r}{\lambda_{q,\nu}(t)}\Big) \to z^* \text{ in } L^2 \text{ as } t \to 0^-$$

with

$$\lambda_{q,\nu}(t)e^{i\gamma_{q,\nu}(t)} = c_\nu \cdot q\frac{|t|^{\frac{\nu}{2}+1}}{|\log|t||}, \qquad \text{in particular} \qquad \lambda_{q,\nu}(t) \sim_{q,\nu} \frac{|t|^{\frac{\mathrm{Re}(\nu)}{2}+1}}{|\log|t||}.$$

Remarks:

- Optimal range of ν.
- When $\mathrm{Im}(\nu) \neq 0$, u exhibits infinite amount of phase rotation.
- Infinite-time blow-up solution by pseudoconformal transform.

Comments on the result

Regularity of z^* and blow-up rate:

- Strong interaction between z^* and $Q_{\lambda,\gamma}$.
- We believe 1-1 correspondence between the blow-up rate and z^*. (This is true for corotational WM: **Jendrej–Lawrie–Rodriguez** '19, whereas it is false for the pseudoconformal blow-up $m \geq 1$)

An interesting part in the proof: construction of the radiation $z(t,r)$.
Evolve the initial data $z(0,r) = qr^{\nu}\chi(r)$ under some NLS (under $i\partial_t + \Delta^{(-2)}$), and justify the leading asymptotics $z(t,r) \approx qc_{\nu}|t|^{\frac{\nu-2}{2}} r^2$ in the self-similar region $r \lesssim |t|^{\frac{1}{2}}$.

Summary of results on blow-up dynamics

In the case of higher equivariance indices $m \geq 1$:

- [1] Constructed a curve of solutions exhibiting the rotational instability of pseudoconformal blow-up solutions.
- [2] Pseudoconformal blow-up solutions can arise from a codimension one set of initial data.
- The results [1,2] motivated the rotational instability conjecture.

In the radial case $m = 0$:

- [3] Constructed smooth finite-energy finite-time blow-up solutions, whose blow-up rate differs from the pseudoconformal one by a power of logarithm. These solutions arise from a codimension one set of initial data. Exhibiting the rotational instability for this case remains open.
- [4] Continuum of blow-up rates.

Comments on the proof

Method of the proof: **modulation analysis**

- [1,4] Backward construction (c.f. Merle '90, Bourgain–Wang '97, Raphaël–Szeftel '11, Merle–Raphaël–Szeftel '13, Jendrej–Lawrie–Rodriguez '19)
- [2,3] Forward construction with repulsivity (c.f. Rodnianski–Sterbenz '10, Raphaël–Rodnianski '12, Merle–Raphaël–Rodnianski '13)

Difficulties:

- Different instability mechanism and evolution equations for b and η.
- **Nonlocal nonlinearity**:
  1. Construction of the blow-up profile P,
  2. Extra phase corrections (due to A_t),
  3. Modified evolution equation for z^*,
  4. Analysis of the linearized operator (from $\partial_t + i\mathcal{L}_Q$ to $\partial_t + iH_Q$)
- These difficulties become the **strongest** in the radial case $m = 0$.
  1. Refined modulation equations,
  2. Weaker repulsivity.

Thank you very much!

Strategy of the proof

The proof is in fact very simple! The key ingredient is the **nonlinear coercivity of energy** as a consequence of **self-duality** and **non-locality**.

- Some standard reductions:
  1. $H^{1,1}$-assumption and pseudoconformal transform
     $\Rightarrow$ suffices to consider the finite-time blow-up case for finite energy solutions.
  2. Finite-time blow-up and blow-up criterion
     $\Rightarrow \|u(t)\|_{\dot{H}^1_m} \to +\infty$ but $E[u(t)]$ is conserved.

  - For defocusing NLS, $E[\psi] \gtrsim \|\nabla\psi(t)\|_{L^2}^2$ so this cannot happen. Thus there is no non-scattering solutions.

Strategy of the proof

Now, the nonlinear coercivity of energy:

- For $m < 0$ (CSS), we show that (CSS) is **defocusing**:

$$E[u] \sim_{M[u]} \|u\|_{\dot{H}^1_m}^2.$$

- For $m \geq 0$ (CSS), we show the **nonlinear coercivity of energy**:

$$E[u] \sim_{M[u]} \|\epsilon^\sharp(t)\|_{\dot{H}^1_m}^2$$

where $u(t) = \frac{e^{i\gamma(t)}}{\lambda(t)} Q\left(\frac{\cdot}{\lambda(t)}\right) + \epsilon^\sharp(t)$ under suitable orthogonality conditions on $\epsilon^\sharp$ and under the small energy regime:

$$0 \leq \frac{E[u(t)]}{\|u(t)\|_{\dot{H}^1_m}^2} \ll 1.$$

This is NOT a linear coercivity, because $\epsilon^\sharp$ **can have very large L^2-norm**.

Combining the above with the arguments of [Merle–Raphaël], soliton resolution follows.

Nonlinear coercivity of energy, $m < 0$

When $m < 0$, our goal is to show

$$E[u] \sim_{M[u]} \|u\|_{\dot{H}^1_m}^2 .$$

To show this, we first use the **self-duality**:

$$E[u] = \frac{1}{2}\int |\mathrm{D}_u u|^2 = \frac{1}{2}\int \Big|\Big(\partial_r - \frac{m + A_\theta[u]}{r}\Big)u\Big|^2 .$$

We recall that $A_\theta[u] = -\frac{1}{2}\int_0^\infty |u|^2 r' dr'$ is always negative with the bound:

$$0 \leq -A_\theta[u] \leq \frac{M[u]}{4\pi}.$$

Note that m is **also negative**. Therefore, we can hope that the linear operator D_u enjoys some Hardy's inequality, and this is in fact true:

$$\int \Big|\Big(\partial_r - \frac{m + A_\theta[u]}{r}\Big)f\Big|^2 \geq C(m, M[u])\|f\|_{\dot{H}^1_m}^2, \qquad \forall f \in \dot{H}^1_m.$$

Point is that C depends only on $M[u]$, which is a conserved quantity. This shows the coercivity $E[u] \gtrsim_{M[u]} \|u\|_{\dot{H}^1_m}^2$.

Nonlinear coercivity of energy, $m \geq 0$

When $m \geq 0$, our goal is equivalent to showing that

$$E[Q+\epsilon] \geq C(m, M[u])\|\epsilon\|_{\dot{H}_m^1}^2, \qquad \forall \epsilon \in \dot{H}_m^1$$

under (i) suitable orthogonality conditions on ϵ and (ii) $\|\epsilon\|_{\dot{H}_m^1} \ll 1$. By **self-duality**,

$$E[Q+\epsilon] = \frac{1}{2}\int |\mathrm{D}_{Q+\epsilon}(Q+\epsilon)|^2 = \frac{1}{2}\int \Big|\Big(\partial_r - \frac{m + A_\theta[Q+\epsilon]}{r}\Big)(Q+\epsilon)\Big|^2.$$

- *Interior* $r \leq R$: (where $\mathrm{D}_{Q+\epsilon}(Q+\epsilon) = L_Q\epsilon + O(\epsilon^2)$)

$$\text{integral} \sim \int_0^R |L_Q\epsilon|^2 r dr.$$

  We can apply the linear coercivity by spending orthogonality conditions.
- *Exterior* $r \geq R$: we use the **non-locality** $A_\theta[Q](r) \to -(2m+2)$ as $r \to +\infty$ to write

$$m + A_\theta[Q+\epsilon] \approx m + A_\theta[Q] + A_\theta[\epsilon] \approx -(m+2) + A_\theta[\epsilon].$$

  and obtain

$$\text{integral} \sim \int_R^\infty \Big|\Big(\partial_r - \frac{-(m+2) + A_\theta[\epsilon]}{r}\Big)\epsilon\Big|^2 r dr$$

  Now $-(m+2)$ is **negative**, prove nonlinear Hardy's inequality as in the $m<0$ case!