On the spatial extent of localized eigenfunctions for random Schrödinger operators

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CY Days 2022
31/03/2022

Lattice Anderson model

Consider $H_\omega = -\Delta + V_\omega$ on $\ell^2(\mathbb{Z}^d)$: $-\Delta\psi(x) = \sum_{|x-y|=1} \psi(y)$ and $V_\omega\psi(x) = \omega_x\psi(x)$

$\omega = (\omega_x)_{x\in\mathbb{Z}^d}$ collection of i.i.d. random variables.

Consider restrictions of H_ω to regions $\Omega \subset \mathbb{Z}^d$ i.e. $(H_\omega)_\Omega = \mathbf{1}_\Omega^T H_\omega \mathbf{1}_\Omega$.

Localization holds in $I \subset \mathbb{R}$ if

$\exists \mu > 0$, $\exists q > 0$ s.t. for L large, with $\mathbb{P} \geq 1 - L^{-p}$, any e.v. $E \in I$ of $(H_\omega)_{|\Lambda_L}$ assoc. to norm. eigenfct φ_E s.t. $\exists x_E \in \Lambda_L$, one has

$$\max_{|x|\leq L} |\varphi_E(x)| e^{\mu|x-x_E|} \lesssim \begin{cases} L^q & \text{in mathematical papers (M)} \\ 1 & \text{in physics papers (P)} \end{cases}$$

Known:

- the bound (P) cannot hold for all eigenfunctions with good probability (Lifshits tails states).
- the bound (M) is optimal (again Lifshits tails states).

Questions: in the localized region, where does the truth lie between (M) and (P)? More precisely,

- how many states satisfy (P)?
- how many states satisfy no estimate "better than (M)"?
- how many states satisfy an estimate "in between (P) and (M)"?

Fix region $\Omega \subset \mathbb{Z}^d$, let $\mathscr{E}((H_\omega)_\Omega) := \{$the set of eigenvalues of $(H_\omega)_\Omega\}$.

Known:

(A1) $\exists \Sigma_p \subset \Sigma \subset \mathbb{R}$ s.t. $\sigma(H_\omega) = \Sigma$ and $\sigma_p(H_\omega) = \Sigma_p$ a.s.

(A2) $\forall E \in \mathbb{R}$, ω-a.s., the integrated density of states exists and is indep. of ω a.s. i.e.

$$N(E) := \lim_{\substack{\Omega \uparrow \mathbb{Z}^d \\ \Omega \text{ finite}}} \frac{\#\mathscr{E}((H_\omega)_\Omega) \cap (-\infty, E]}{\#\Omega} \tag{1}$$

To $\varphi \in \ell^2(\Omega)$, with $\Omega \subset \mathbb{Z}^d$, we associate the *set of localization centers*:

$$\mathscr{C}(\varphi) := \{x \in \Omega : |\varphi(x)| = \|\varphi\|_\infty\}. \tag{2}$$

(A3) $\exists I_{\mathrm{AL}} \subseteq \Sigma$, a union of finitely many open intervals,s.t. for any region Ω, with probability one, $(H_\omega)_\Omega$ has pure point spectrum on I_{AL}.Furthermore, there are constants $A_{\mathrm{AL}} < \infty$, $\mu > 0$ s.t. if $\Omega \subset \mathbb{Z}^d$ is a region, $S \subset \Omega$ is a finite set, and $0 < \varepsilon < 1$,then, with probability larger than $1 - \varepsilon$, any ℓ^2-normalized eigenfunction φ_E of $(H_\omega)_\Omega$ with eigenvalue $E \in I_{\mathrm{AL}} \cap \mathscr{E}((H_\omega)_\Omega)$ and $\mathscr{C}(\varphi_E) \cap S \neq \emptyset$ satisfies

$$\max_{y \in \mathscr{C}(\varphi_E) \cap S} \left(\sum_{x \in \Omega} e^{2\mu|x-y|} |\varphi_E(x)|^2 \right)^{\frac{1}{2}} \leq A_{\mathrm{AL}} \left(\frac{\#S}{\varepsilon}\right)^{\frac{1}{2}}. \tag{3}$$

Given $\mu > 0$, define a decreasing family of weighted ℓ^2-norms by

$$M_\ell^\mu(\varphi; y) := \left(\sum_{x \in \Omega} e^{2\mu(|x-y|-\ell)_+} |\varphi(x)|^2 \right)^{\frac{1}{2}} \quad \text{where } \ell = 0, 1, 2, \ldots \tag{4}$$

If $M_\ell^\mu(\varphi; y)$ is finite, then $\lim_{\ell \to \infty} M_\ell^\mu(\varphi; y) = \|\varphi\|_{\ell^2}$.

Onset length: $\ell_\mu(\varphi; y) := \min\{\ell : M_\ell^\mu(\varphi; y) \leq 2\|\varphi\|_2\}$.

Proposition

Let $\Omega \subset \mathbb{Z}^d$ be region and $S \subset \Omega$ a finite set. If $0 < \varepsilon < 1$, then, with probability larger than $1 - \varepsilon$, any ℓ^2-normalized eigenfunction φ_E of $(H_\omega)_\Omega$ with eigenvalue $E \in I_{\mathrm{AL}} \cap \mathscr{E}((H_\omega)_\Omega)$ and $\mathscr{C}(\varphi_E) \cap S \neq \emptyset$ satisfies

$$\left(\sum_{x \in \Omega} e^{2\mu(|x-x_E|-\ell_E)_+} |\varphi_E(x)|^2 \right)^{\frac{1}{2}} \leq 2, \quad \textit{for any } x_E \in \mathscr{C}(\varphi_E) \cap S \tag{5}$$

with onset length $\ell_E = \ell_\mu(\varphi_E; x_E) < \frac{1}{\mu}\left(\log A_{\mathrm{AL}} + \frac{1}{2}\log \#S - \frac{1}{2}\log 3\varepsilon\right) + 1$.

Given two localization centers $x_E, x'_E \in \mathscr{C}(\varphi_E)$ i.e.

Proposition

$$|\ell_\mu(\varphi_E; x_E) - \ell_\mu(\varphi_E; x'_E)| \leq |x_E - x'_E| \tag{6}$$

The onset length $\ell_\mu(\phi_E; x_E)$ also gives an upper bound on the diameter of $\mathscr{C}(\varphi_E)$, namely,

Proposition

Pick $\kappa > 0$ such that $8e^{-2\mu\kappa} = 1$. If $(x_E, x'_E) \in \mathscr{C}(\varphi_E)^2$ then

$$|x_E - x'_E| \leq \ell_\mu(\varphi_E; x_E) + \frac{d}{2\mu} \log\left(2\ell_\mu(\varphi_E; x_E) + 2\kappa + 1\right) + \frac{3\log 2}{2\mu} \tag{7}$$

This yields pointwise bound $|\varphi_E(x)| \leq \|\varphi_E\|_\infty e^{-\mu(|x-x_E|-\tilde{\ell}_{\mu,E})_+}$ where

$$\tilde{\ell}_{\mu,E} = \ell_\mu(\varphi_E; x_E) + \tfrac{d}{2}\log(2\ell_\mu(\varphi_E; x_E) + 2\kappa + 1)$$

Assume

(A4) $\exists A_M > 0$ s.T., for any finite region $\Omega \subset \mathbb{Z}^d$, one has

$$\sup_{E \in I_{\mathrm{AL}}} \mathbb{P}\left(\left\{\mathrm{tr}(\mathbf{1}_{[E-\varepsilon, E+\varepsilon]}((H_\omega)_\Omega)) \geq 2\right\}\right) \leq A_M |\Omega|^2 \varepsilon^2 \tag{8}$$

for any $\varepsilon > 0$.

Main results quantify the distribution of onset lengths for eigenfunctions with localization centers in a given bounded region.

Set $\Lambda = \Lambda_L(x_0) = \left(x_0 + \left]-\frac{L}{2}, \frac{L}{2}\right]^d\right) \cap \mathbb{Z}^d$.

Theorem

Let μ and I_{AL} be as in (A3) and let $[a,b] \subset I_{\mathrm{AL}}$. Then, for any $\nu < \mu$ and any $p > 0$ there exist $C_\nu > 0$, $\ell_0 > 0$, and $L_0 > 0$ such that if $\Omega \subset \mathbb{Z}^d$ is a region with $\Lambda \subset \Omega$ with $L \geq L_0$, then with probability larger than $1 - L^{-p}$, for all $\ell \geq \ell_0$, one has

$$\#\{E \in \mathscr{E}((H_\omega)_\Omega) \cap [a,b] \text{ s.t. } \exists x_E \in \mathscr{C}(\varphi_E) \cap \Lambda \text{ and } \ell_\nu(\varphi_E; x_E) \geq \ell\} \leq L^d e^{-C_\nu \ell}. \tag{9}$$

Theorem

Let μ and I_{AL} be as in (A3) and let $[a,b] \subset I_{\mathrm{AL}}$. Pick $\alpha > d$. Then, for any $\nu < \mu$, there exists $c > 0$ s.t. for any $\ell \geq 1$, one has

$$\sup_{x\in\Omega} \mathbb{E}\left[\sum_{E\in\mathscr{E}((H_\omega)_\Omega)\cap[a,b]} \sum_{\substack{x_E\in\mathscr{C}(\varphi_E)\ s.t\\ \ell_\nu(\varphi_E;x_E)\geq\ell}} \left(\frac{1}{1+\frac{|x-x_E|}{\ell_\nu(\varphi_E;x_E)}}\right)^\alpha\right] \leq \frac{1}{c}e^{-c\ell}\ell^{d+1}. \qquad (10)$$

Thus, the localization centers corresponding to large onset length are more or less uniformly distributed and quite remote from one another.

Corollary

For $\nu < \mu$, $a < b$ real such that $[a,b] \subset I_{\mathrm{AL}}$, and $\ell > 0$, the limit

$$N_\nu([a,b],\ell) := \lim_{L\to+\infty} \frac{\#\{E \in \mathscr{E}(H_\omega)\cap[a,b] \ : \ \exists \mathbf{x}_E \in \mathscr{C}(\varphi_E)\cap\Lambda \ s.t.\ \ell_\nu(\varphi_E;x_E) \geq \ell\}}{N(I)\cdot L^d}$$

exists a.s. and is independent of ω.

Optimality of the upper bound on the counting function: a lower bound

One has the deterministic lower bound

Theorem

Let μ and I_{AL} be as in (A3). Let E_- be the infimum of Σ the almost sure spectrum of H_ω and assume $E_- > -\infty$. Then, $\exists c > 0$, $\ell_0 > 0$ and $L_0 > 0$ s.t. for any $\nu < \mu$, for $\Lambda = \Lambda_L$ with $L \geq L_0$, and all $\ell \geq \ell_0$, with probability 1, one has

$$\#\{E \in \mathscr{E}(H_\omega) \ : \exists E \in \mathscr{C}(\varphi_E)\cap\Lambda \neq \emptyset \ s.t.\ \ell_\nu(\varphi_E;x_E) \geq \ell\}$$
$$\geq \#\{E \in \mathscr{E}(H_\omega)\cap[E_-, E_- + c\ell^{-d-1}] \ : \ \mathscr{C}(\varphi_E)\cap\Lambda \neq \emptyset\}. \qquad (11)$$

Corollary

Let E_- be the infimum of Σ the almost sure spectrum of H_ω and assume $E_- > -\infty$. Then there exists $\ell_0 > 0$ such that, for any $\nu < \mu$ and $\ell \geq \ell_0$, one has

$$N_\nu(\Sigma,\ell) \geq N(E_- + c\ell^{-d-1}) \qquad (12)$$

Known (Lifshits tails): $N(E_- + \lambda) \geq e^{-f(\lambda)\lambda^{-d/2}}$.

Numerics: some eigenfunctions

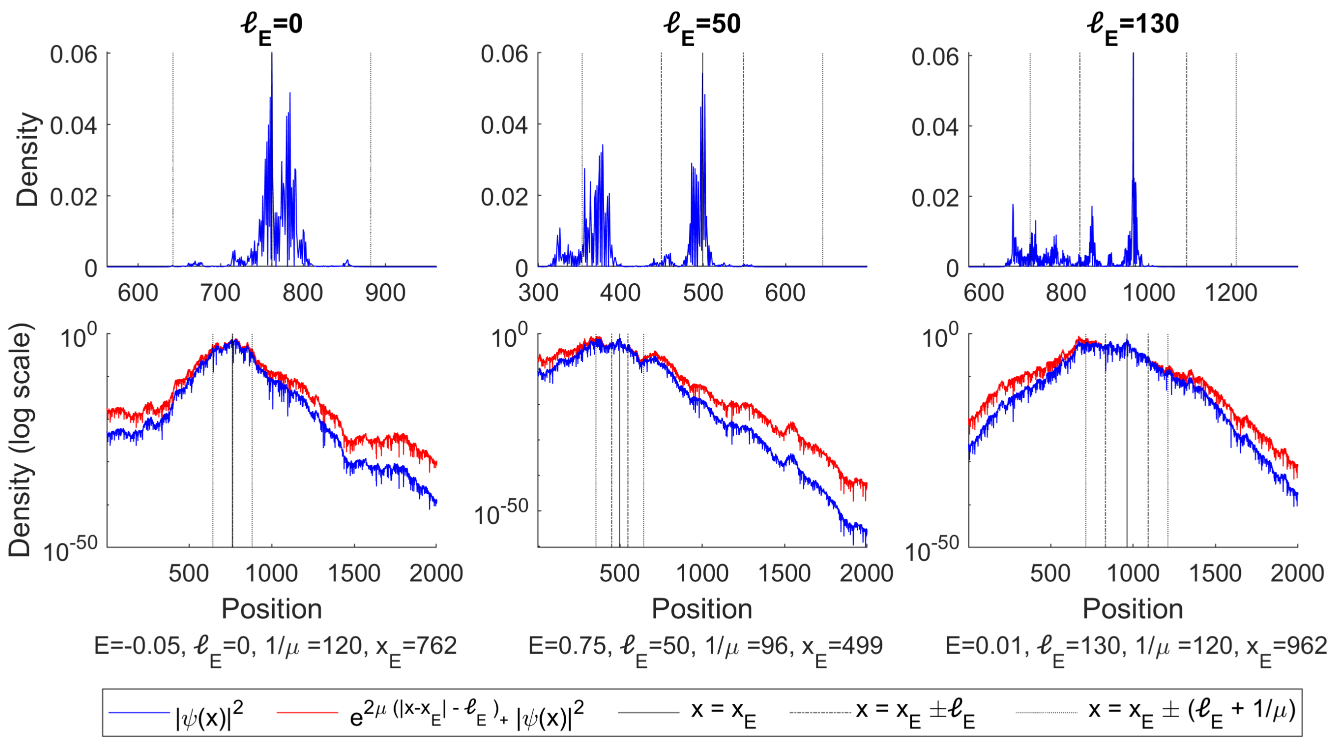

Figure: Three eigenfunctions for the 1D Anderson model on $\Lambda = [1, 2000]$ with $\lambda = 1$.

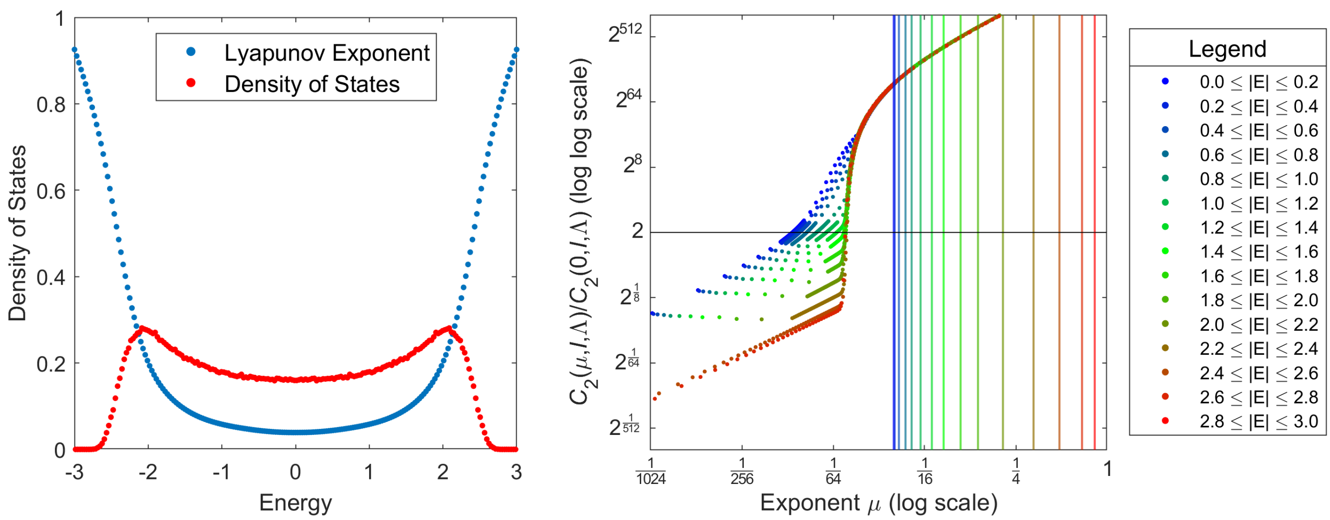

Figure: (a) Lyapunov Exponent, Density of States (b) Correlators for an interval of length 2000 for the 1D Anderson model with disorder $\lambda = 1$.

(a) Lyapunov exponent and density of states versus energy

(b) Normalized eigenfunction correlators $\frac{C_2(\mu,I,\Lambda)}{C_2(0,I,\Lambda)}$ vs. exponent μ for the energy intervals shown. The Lyapunov exponent $L(I)$ for each interval is indicated as a vertical line. Here

$$C_2(\mu,I,\Lambda) = \frac{1}{\#\Lambda}\sum_{x,y\in\Lambda} e^{2\mu|x-y|}\mathbb{E}\left(\sum_{E\in I\cap\mathscr{E}((H_\omega)_\Lambda)} |\varphi_E(x)|^2|\varphi_E(y)|^2\right). \quad (13)$$

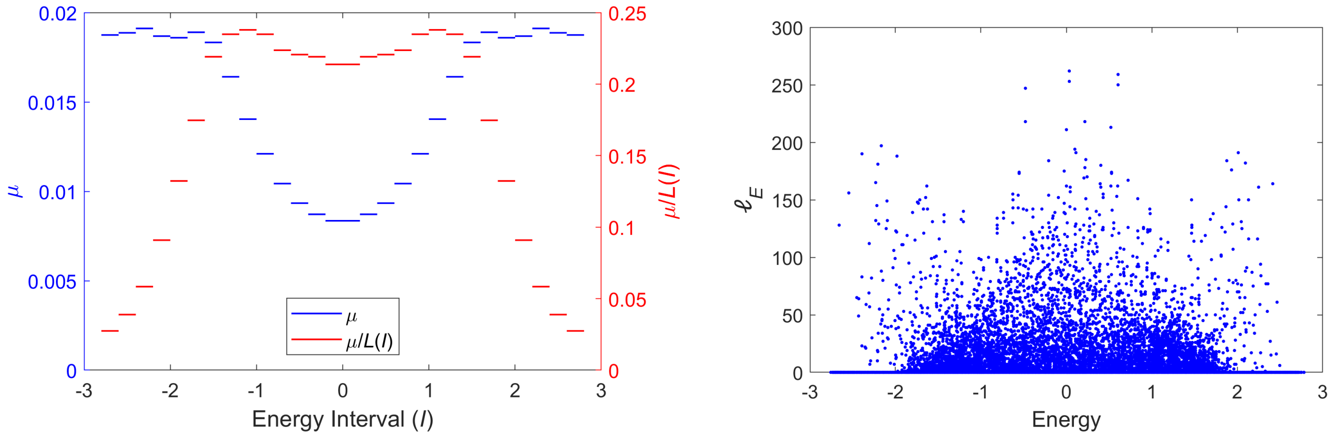

Figure: Exponents (a) and onset length (b) for eigenfunctions of the 1D Anderson model on an interval of 2000 with disorder $\lambda = 1$.

(a) On each energy interval with observed eigenvalues, the exponent μ at which $\dfrac{C_2(\mu,I,\Lambda)}{C_2(0,I,\Lambda)} = 2$ is shown in blue and the ratio $\frac{\mu}{L(I)}$ is plotted in red, where $L(I)$ is the minimal Lyapunov exponent on I.

(b) Onset length ℓ_E versus energy E for the eigenfunctions from 240 samples ($480,000$ eigenfunctions in total). Only $6,313$ (1.3% of the total) eigenfunctions have $\ell_E > 0$.

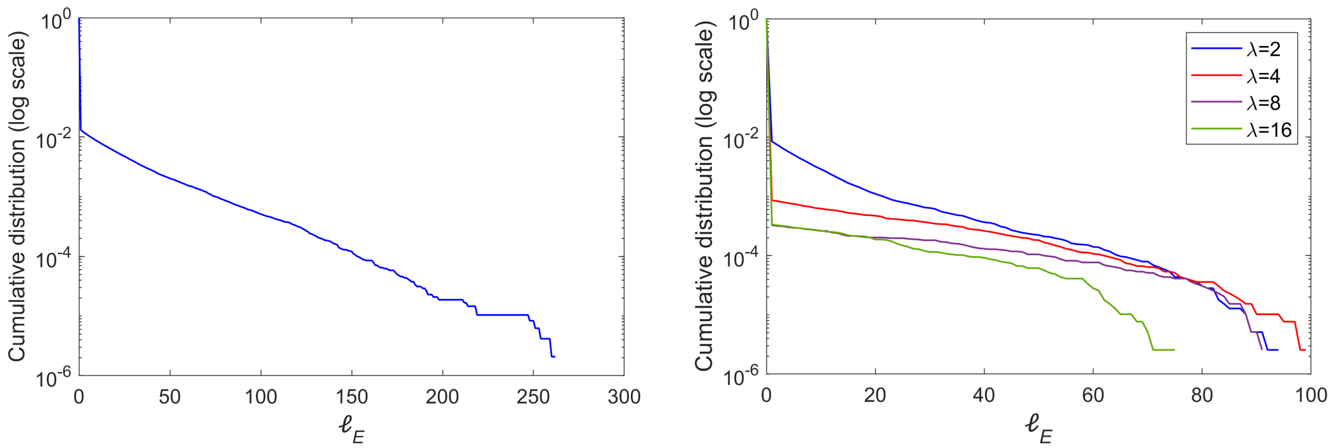

Figure: Cumulative distributions

(a) Cumulative distribution of onset lengths for eigenfunctions of the 1D Anderson model.

(b) Cumulative distribution of onset lengths for 240 samples with $\lambda = 1$ on an interval of length 2000.

Numerics show onset lengths as large as 262.
Question: consistency with the estimation that onset lengths "of order $\log \#\Lambda = \log 2000 \approx 7.6$."?
Correlator bound $C_2(\mu, I, \Lambda) \leq 2 * C_2(0, I, \lambda)$ provides an *a priori* bound on localization lengths consistent with this observation.
Indeed, from the Markov inequality, with proba $\geq 1-\varepsilon$,

$$\frac{1}{\#\Lambda}\sum_{x,y\in\Lambda} e^{2\mu|x-y|} \sum_{E\in I\cap\mathscr{E}((H_\omega)_\Lambda)} |\varphi_E(x)|^2|\varphi_E(y)|^2 \leq \frac{2C_2(0,I,\lambda)}{\varepsilon}.$$

Taking $y = x_E$ (E fixed), each eigenfunction satisfies

$$\sum_x e^{2\mu|x-x_E|}|\varphi_E(x)|^2 \leq \frac{2\,C_2(0,I,\Lambda)\,\#\Lambda}{\varepsilon\|\varphi_E\|_\infty^2}.$$

First proposition implies that

$$\ell_E \leq \frac{1}{2\mu}(\log 2C_2(0,I,\Lambda) + \log\#\Lambda - \log 3\varepsilon - 2\log\|\varphi_E\|_\infty) + 1. \tag{14}$$

Numerical context: $\varepsilon = 1/240$ (as 240 samples).
Key point: onset lengths to be no larger than $\frac{1}{2\mu}(\log\#\Lambda - \log 3\varepsilon)$.
$\#\Lambda = 2000$, $\varepsilon = 1/240$, and $\mu \approx 0.01 \Longrightarrow$ rough bound of order 600.