ENTIRE VORTEX SOLUTIONS OF NEGATIVE DEGREE FOR THE ANISOTROPIC GINZBURG-LANDAU SYSTEM

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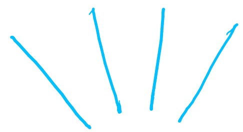
ANISOTROPIC GINZBURG LANDAU SYSTEM

$$\Delta u + \delta\left(\nabla(\nabla\cdot u) - \nabla^{\perp}(\nabla\times u)\right) = (|u|^2-1)u$$

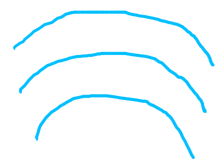
$u : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \delta \in (-1,1)$

ASSOCIATED WITH ANISOTROPIC GL ENERGY:

$$E_\delta(u;\Omega) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{\delta}{2}\left((\nabla\cdot u)^2 - (\nabla\times u)^2\right) + \frac{1}{4}(1-|u|^2)^2$$



SPLAY



BEND

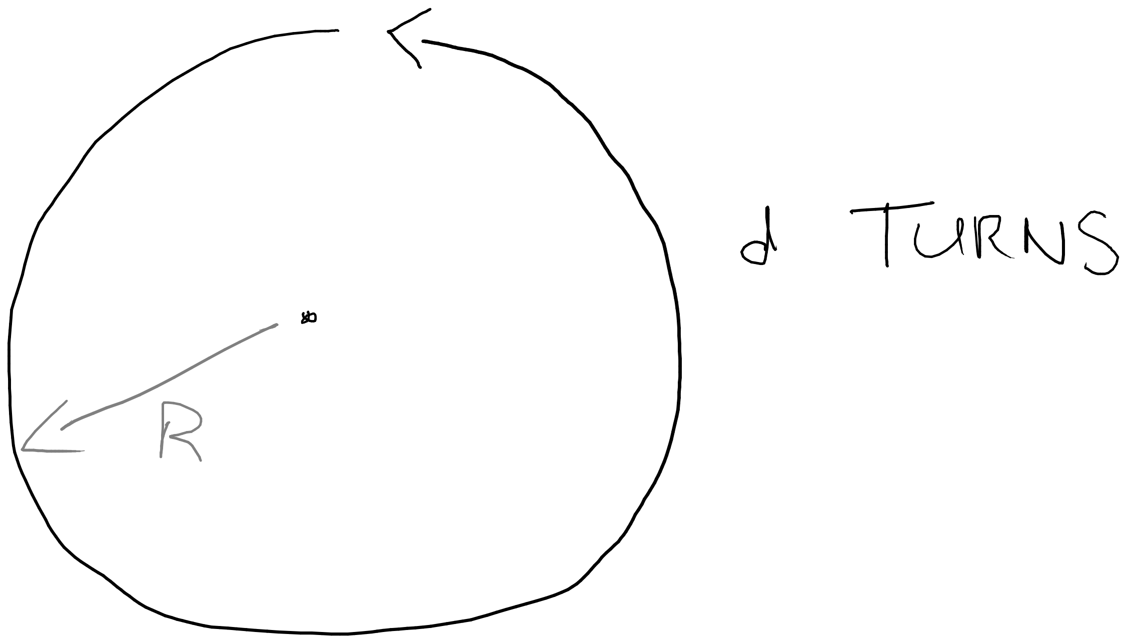
(LIQUID CRYSTALS ...)

QUESTION: EXISTENCE OF SOLUTIONS $u:\mathbb{R}^2 \to \mathbb{R}^2$ WITH PRESCRIBED DEGREE $d \in \mathbb{Z}$:

$$\int_{\mathbb{R}^2} (1-|u|^2)^2 < \infty \quad , \quad \deg(u, \partial D_R) = d \quad \forall R \gg 1$$

I.E. $u(Re^{i\theta}) = |u|\, e^{i\varphi_R(\theta)}$, $\varphi_R(2\pi) - \varphi_R(0) = 2\pi d$

"DEGREE d VORTEX"



ISOTROPIC CASE $\boxed{\delta = 0}$:

$$\Delta u = (1-|u|^2)u$$

↳ FOR ALL $d \in \mathbb{Z}$ THERE EXIST SOLUTIONS

$$u(re^{i\theta}) = f_d(r)\, e^{i(\theta_0 + d\theta)}$$

$f_d : [0,\infty) \to \mathbb{R}$, $f_d(\infty) = 1$, ODE [Hervé-Hervé '94]

[Chen-Elliott-Qi '94]

ANISOTROPIC CASE $\boxed{\delta \neq 0}$: SOLUTIONS OF THIS FORM EXIST ONLY FOR

$d = 1$

$\theta_0 \in \pi/2\, \mathbb{Z}$

↳ CANNOT REDUCE TO 1D PROBLEM FOR $d \neq 1$

THEOREM. FOR $d \leq -1$ AND $|\delta| \leq \delta_0(d)$,

THERE EXIST SOLUTIONS $u : \mathbb{R}^2 \to \mathbb{R}^2$ OF

$$\Delta u + \delta\left(\nabla(\nabla \cdot u) - \nabla^\perp(\nabla \times u)\right) = (|u|^2 - 1)u$$

$$\int_{\mathbb{R}^2} (1-|u|^2)^2 < \infty, \quad \deg(u) = d$$

→ IMPORTANT PHYSICAL CASE $d = -1$

→ DEGREE CONDITION IMPOSES

$$E_\delta(u; D_R) \geq \pi(1-|\delta|)d^2 \ln R + O(1) \qquad R \gg 1$$

CANNOT MINIMIZE DIRECTLY IN $\mathbb{R}^2$

↳ MINIMIZE IN D_R AND $R \to \infty$: CONSERVE DEGREE ?

PROOF IN 2 STEPS:

1. MINIMIZATION IN A SYMMETRIC CLASS WHICH IMPOSES

$$\deg(u) = D \in \underbrace{d + 2(1-d)\mathbb{Z}}_{\text{DOESN'T CONTAIN } 0}$$

[OK $\delta \in (-1,1)$]

2. FOR $|\delta|$ SMALL, MINIMALITY IMPOSES

$$|D| = \min \{ |d'| : d' \in d + 2(1-d)\mathbb{Z} \}$$

$$\Rightarrow \quad D = d$$

[DOESN'T WORK IF $d \geq 2$]

1 SYMMETRIES

$$u(z) \longrightarrow e^{-i\alpha} u(e^{i\alpha} z) \qquad \alpha \in \mathbb{R}$$

$$u(z) \longrightarrow -u(z)$$

LEAVE ENERGY INVARIANT

ISOTROPIC CASE $\delta = 0$: MORE SYMMETRIES

$$u(z) \longrightarrow u(e^{i\alpha} z) \qquad \alpha \in \mathbb{R}$$

$$u(z) \longrightarrow e^{i\beta} u(z) \qquad \beta \in \mathbb{R}$$

(ONLY $\alpha, \beta \in \pi\mathbb{Z}$ IF $\delta \neq 0$)

EXAMPLE: FOR $\delta = 0$, FUNCTIONS

INVARIANT UNDER

$$\mu(z) \longrightarrow e^{-id\alpha}\mu(e^{i\alpha}z) \qquad \alpha \in \mathbb{R}$$

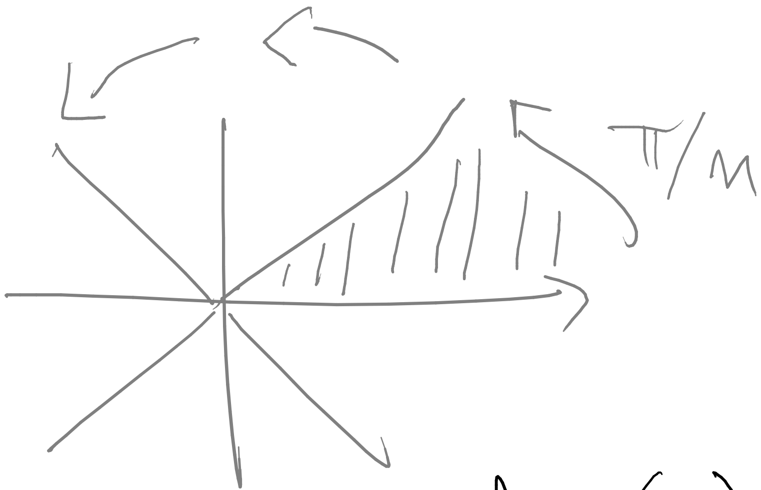
HAVE THE FORM

$$\mu(re^{i\theta}) = f(r)\, e^{i(\theta_0 + d\theta)}$$

(CANNOT USE THESE TRANSFORMATIONS IF $\delta \neq 0$ AND $d \neq 1$)

HERE WE USE A FINITE GROUP OF SYMMETRIES, GENERATED BY

$$u(z) \longrightarrow -e^{-i\pi/m}\,u(e^{i\pi/m}z) \qquad (m \neq 0 \text{ Fixed})$$



SYMMETRIC FUNCTIONS WITH $\int (1-|u|^2)^2 < \infty$ MUST HAVE DEGREE

$$\deg(u) \equiv (1-m) \text{ MODULO } 2m$$

$\Leftrightarrow$ CHOOSE $m = 1-d$

PROPOSITION: FOR $|\delta|<1$, $\exists\, u:\mathbb{R}^2\to\mathbb{R}^2$ SYMMETRIC SOLUTION WITH $\int_{\mathbb{R}^2}(1-|u|^2)^2<\infty$

AND $E_\delta(u;D_R)\leq E_\delta(v;D_R)$ FOR ALL v SYMMETRIC S.T. $v_{\lfloor \partial D_R}=u_{\lfloor \partial D_R}$

$\Longleftrightarrow \deg(u)=D\in d+2(1-d)\mathbb{Z}$

IDEAS:

1. MINIMIZE

$$E(u_R; D_R) = \min \left\{ E(u^{sym}; D_R) : u^{sym}\lfloor_{\partial D_R} = \xi_R \right\}$$

FOR WELL-CHOSEN ξ_R $\left(\text{MIN } E(\cdot\,; \partial D_R) \right)$

$$u = \lim_{R\to\infty} u_R \quad \text{in } C^2_{loc}(\mathbb{R}^2) \qquad \text{NONTRIVIAL ?}$$

2. POHOZAEV IDENTITY (EQ $\times\, \partial_r u$)

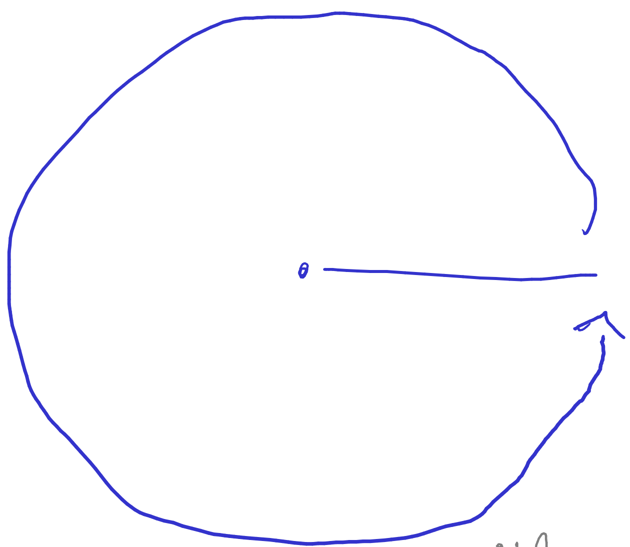
$$\frac{1}{2}\int_{D_R} (1-|u_R|^2)^2 = R\, E(\xi_R; \partial D_R) - R\int_{\partial D_R} \underbrace{\text{funct}^{\circ}(\partial_r u_R)}_{\geq 0}$$

$$\rightsquigarrow \int_{D_R} (1-|u_R|^2)^2 \leq C \implies \int_{\mathbb{R}^2} (1-|u|^2)^2 < \infty$$

② ENERGY BOUNDS $\Rightarrow$ $D = d$

- LOWER BOUND

$$\frac{E_\delta(u; D_R)}{\pi \ln R} \geq (1-\delta) D^2 + o(1) \qquad R \gg 1$$



$u \approx e^{i\varphi} \quad (r \gg 1)$

STANDARD IDEA

$$E_\delta(e^{i\varphi}) \geq \frac{1-\delta}{2} \int |\nabla \varphi|^2$$

$$\geq \frac{1-\delta}{2} \int_{R_0}^{R} \int_0^{2\pi} \frac{(\partial_\theta \varphi)^2}{r^2} \, r \, dr$$

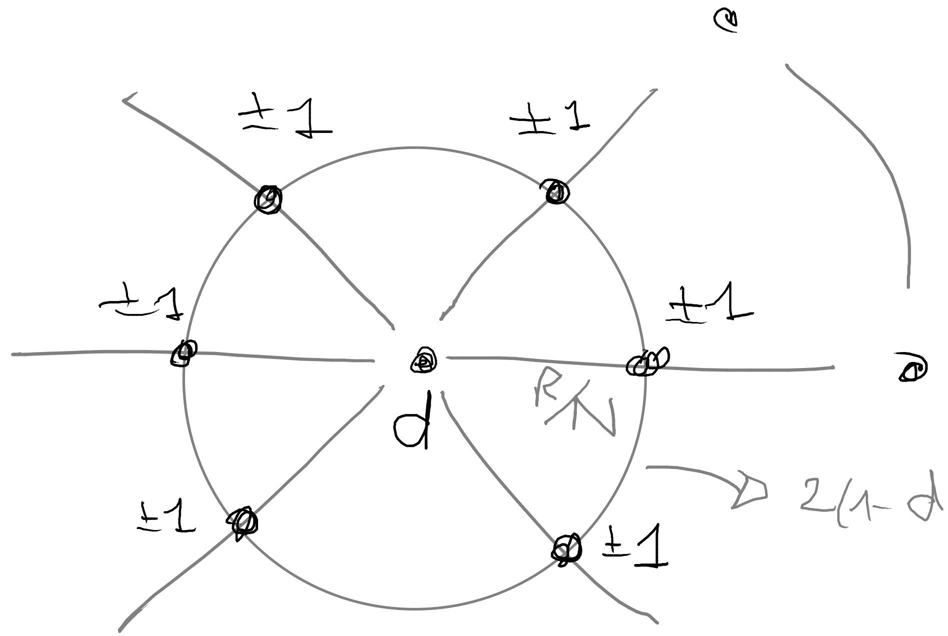
$$\geq \frac{1-\delta}{2} \int_{R_0}^{R} \frac{1}{2\pi} \left(\int_0^{2\pi} \partial_\theta \varphi \right)^2 \frac{dr}{r}$$

$$= \pi (1-\delta) D^2 \ln \frac{R}{R_0}$$

• UPPER BOUND

$$\frac{E_\delta(u; D_R)}{\pi \ln R} \le (1+3\delta)\Big(d^2 + \underbrace{|D-d|}_{=2(1-d)N}\Big) + o(1)$$

SYMMETRIC COMPETITOR



$\to$ $2(1-d)$ vortices of ~~degree~~ ± 1

[SHAFRIR '94]

TECHNICAL DIFFICULTY: NEED A PRIORI BOUND

$$E_\delta(u; D_R) \lesssim \ln R \qquad R \gg 1$$

TO PERFORM THE UPPER BOUND CONSTRUCTION

$\hookrightarrow \delta = 0$ BREZIS-MERLE-RIVIERE '94

($\int \frac{|\partial_\theta u|^2}{r^2}$ DOMINATES ["MONOTONICITY FORMULA"])

$\hookrightarrow$ HERE NEED TO USE MINIMALITY AND ANOTHER COMPETITOR CONSTRUCTION

A PRIORI LOGARITHMIC BOUND $E(u; D_R) \lesssim \ln R$

(1) $u = \rho e^{i(D\theta + \varphi)}$ on ∂D_R

$\hookrightarrow$ smooth extensions of ρ, φ:

$$F(R) = E(u; D_R) \leq c_0 R\, E(u; \partial D_R) + c_1 \ln R$$
$$= c_0 R F'(R) + c_1 \ln R$$

$\Rightarrow$ $F \lesssim \ln R$ OR $F \gtrsim R^{\frac{1}{c_0}}$

(2) BASIC ENERGY BOUND $F \lesssim R$ (EQ $\times u$)

(3) OBTAIN $c_0 < 1$ (POHOZAEV & HARMONIC EXTENSIONS)

LOWER & UPPER BOUND

$$(1-\delta) D^2 \leq (1+3\delta)\left(d^2 + |D-d|\right)$$

$$\Longrightarrow \quad D = d \quad \text{if } |\delta| < \delta_0(d) \quad (\text{EXPLICIT})$$

$$\left(\begin{array}{l} \left.\begin{array}{l} \delta = 0: \quad D^2 \leq d^2 + |D-d| \\ \& \quad D \in d + 2(1-d)\mathbb{Z} \end{array}\right\} \Longrightarrow D = d \\ \text{BECAUSE } |D| \geq |d| \end{array} \right)$$

OPEN QUESTIONS:

* SYMMETRIC SOLUTION EXISTS FOR ALL $\delta \in (-1,1)$: WHAT HAPPENS TO ITS DEGREE D FOR $|\delta|$ NOT SMALL?

* EXISTENCE OF DEGREE -1 VORTEX FOR ALL $\delta \in (-1,1)$?

* STABILITY OF DEGREE -1 VORTEX ?

THANK YOU

FOR YOUR

ATTENTION !