Probabilistic well-posedness for a nonlinear Grushin-Schrödinger equation

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March 2022, CY Days in Nonlinear Analysis

The work presented has been done in collaboration with Louise Gassot (University of Basel)

Motivation for the Grushin NLS

Consider $u(t) : \mathbb{R}^2 \longrightarrow \mathbb{C}$ satisfying

$$\begin{cases} i\partial_t u + \Delta u &= |u|^2 u \\ u(0) &= u_0 \in H^s(\mathbb{R}^2)\,. \end{cases} \tag{NLS}$$

Two formal conserved quantities:

$$M(t) = \|u(t)\|_{L^2}^2 \text{ and } E(t) = \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + \frac{1}{4}\|u(t)\|_{L^4}^4\,.$$

- Scaling $s_c = 0$. Standard LWP for $s > 0$ uses dispersion: Cazenave-Weissler '90,
- Global theory in H^1, even in L^2 Dodson '16.

Motivation for the Grushin NLS

What if we change a little bit the setting? Consider the equation

$$\begin{cases} i\partial_t u + \Delta_G u &= |u|^2 u \\ u(0) &= u_0 \in H_G^k , \end{cases} \tag{NLS-G}$$

$-\Delta_G = -\partial_x^2 - x^2\partial_y^2$ is the Grushin operator.

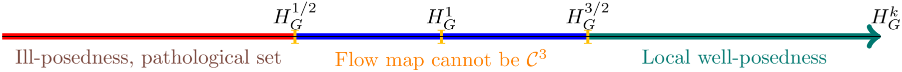
$\|u\|_{H_G^k} = \|\langle -\Delta_G \rangle^{\frac{k}{2}} u\|_{L^2}$ are the adapted Sobolev spaces.

$$u \mapsto u_\lambda(t,x) = \lambda u(\lambda^2 t, \lambda x, \lambda^2 y)\,,$$

scaling invariance $H_G^{\frac{1}{2}}$ critical.

Question: Can we construct solutions in H_G^k for $k > \frac{1}{2}$?

Deterministic picture



Well-posedness part $k > 3/2$: Bahouri-Gallagher '01, stated for NLS-$\mathbb{H}^1$.

Proposition (Best local theory : Bahouri-Gallagher '01)

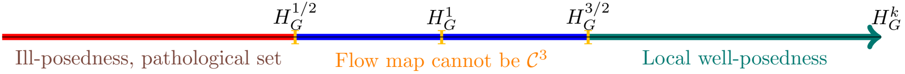
The Cauchy problem for (NLS-G) *is locally well-posed in* $\mathcal{C}^0([0,T], H_G^k)$ *as soon as* $k > \frac{3}{2}$.

Consequence of Sobolev embedding $H_G^{\frac{3}{2}^+} \hookrightarrow L^\infty$.

No result is known at energy regularity in H_G^1!

Bahouri-Barilari-Gallagher '19: anisotropic Strichartz estimates on $\|e^{it\Delta_G} u_0\|_{L_y^\infty L_T^p L_x^q}$. Does not lead to a better local theory.

Deterministic picture



- $1/2 < k < 3/2$ part due to Bahouri-Gérard-Xu '00: due to non-existence of Strichartz estimate $L^4_{t,x,y}$ (See Gérard-Grellier '10 and Remark 2.12 in Burq-Gérard-Tzvetkov '04).
- Ill-posedness part $k < 1/2$: Camps-Gassot '22. G_δ dense set of initial data producing norm inflation: for example, data $\|u_0\|_{H^k_G} \sim 1$ and $\|u(\varepsilon)\|_{H^k} > 2$ for $\varepsilon \ll 1$.

  Uses the supercriticality of the scaling!

Main result: random data techniques

Theorem (Deterministic version)

Let $k \in (1, \frac{3}{2}]$ There exists a dense set $X \subset H^k_G \setminus \bigcup_{\varepsilon>0} H^{k+\varepsilon}_G$ such that for every $u_0 \in X$ there exists a unique solution to (G-NLS) associated to u_0 in the space $e^{it\Delta} u_0 + \mathcal{C}^0([0, T_{u_0}], H_G^{\frac{3}{2}^+}) \hookrightarrow \mathcal{C}^0([0, T_{u_0}], H^k_G)$.

Theorem (Probabilistic version)

Let $k \in (1, \frac{3}{2}]$. There exists a measure μ_k supported by $\mathcal{X}^k_1 \subset H^k_G \setminus \bigcup_{\varepsilon>0} H^{k+\varepsilon}_G$ such that for μ_k almost-every $u_0 \in \mathcal{X}^k_1$ a unique solution to (G-NLS) exists in the space $u_L + \mathcal{C}^0([0, T], H_G^{\frac{3}{2}^+})$, where $u_L(t) = e^{it\Delta_G} u_0$.

We can construct many such measures μ_k whose set $\mathcal{M}_k$ satisfies:

$$\overline{\bigcup_{\mu_k \in \mathcal{M}_k} \operatorname{supp} \mu_k} = H^k_G \setminus \bigcup_{\varepsilon>0} H^{k+\varepsilon}_G \,.$$

What are we going to do?

1. **The probabilistic local well-posedness framework**
2. The randomisation
3. The linear random estimates: first Cauchy theory
4. The bilinear random estimate: better Cauchy theory
5. Random-deterministic bilinear estimates

General probabilistic well-posedness framework

$$\partial_t u + \Delta_G u = |u|^2 u \qquad \text{(NLS-G)}$$

The goal is to seek for $u = u_L^\omega + v$ solution to (NLS-G) (Bourgain '96, Burq-Tzvetkov '08) where:

- u_L^ω has regularity H^k, but is explicit and benefits from improved random estimates (more on that later!).
- v is more regular, *i.e.*, the deterministic regularity $H_G^{\frac{3}{2}^+}$ which solves

$$i\partial_t v + \Delta_G v = |u_L^\omega + v|^2 (u_L^\omega + v)\,.$$

Only need to solve the equation in v, requires estimates on u_L^ω!

Fixed point argument

Fixed point on the Duhamel formulation in v:

$$v(t) = \Phi(v)(t) := -i \int_0^t e^{i(t-t')\Delta_G}(|u_L^\omega + v|^2(u_L^\omega + v))\,\mathrm{d}t'\,,$$

We bound

$$\begin{aligned}\|\Phi v\|_{L_T^\infty H_G^{\frac{3}{2}+}} &\leqslant \int_0^T \||u_L^\omega + v|^2(u_L^\omega + v)\|_{H_G^{\frac{3}{2}+}} \\ &\lesssim \||u_L^\omega|^2 u_L^\omega\|_{L_T^1 H_G^{\frac{3}{2}+}} + \|u_L^\omega |v|^2\|_{L_T^1 H_G^{\frac{3}{2}+}} + \||u_L^\omega|^2 v\|_{L_T^1 H_G^{\frac{3}{2}+}} + \||v|^2 v\|_{L_T^1 H_G^{\frac{3}{2}+}} \\ &\quad + \{\text{similar terms}\}\,.\end{aligned}$$

Identifying problematic terms

- The bound $\||v|^2 v\|_{L^1_T H_G^{\frac{3}{2}+}} \lesssim T\|v\|^3_{L^\infty_T H_G^{\frac{3}{2}+}}$ is obtained by composition and Sobolev embedding.
- For $|u_L^\omega|^2 u_L^\omega$ the problem is that u_L^ω is not even $H_G^{\frac{3}{2}^+}$. Some non-trivial smoothing is needed!
- For $u_L^\omega |v|^2$ Also non-trivial smoothing needed: more complicated

1. The probabilistic local well-posedness framework ✔
2. **The randomisation**
3. The linear random estimates: first Cauchy theory
4. The bilinear random estimate: second Cauchy theory
5. The mixed terms random-deterministic?

Fourier decomposition

Fourier partial transform in y of $-\Delta_G = -\partial_x^2 - x^2\partial_y^2$,

$$\mathcal{F}_{y\to\eta}(-\Delta) = -\partial_x^2 + |\eta|^2 x^2 \,,$$

which is a rescaled harmonic oscillator of L^2-normalised eigenfunctions $|\eta|^{\frac{1}{4}} h_m(|\eta|^{\frac{1}{2}}\cdot)$ associated to eigenvalue $(2m+1)|\eta|$where the h_m are Hermite functions, $(-\Delta + |x|^2)h_m = (2m+1)h_m$.

Any $u \in L^2_G$ thus writes

$$\mathcal{F}_{y\to\eta}u(x;\eta) = \sum_{m\geqslant 0} f_m(\eta) h_m(|\eta|^{\frac{1}{2}}x) = \sum_{m\geqslant 0} \frac{f_m(\eta)}{|\eta|^{1/4}} |\eta|^{\frac{1}{4}} h_m(|\eta|^{\frac{1}{2}}x) \,,$$

and we have

$$\|u\|_{H^k}^2 = \sum_{m\geqslant 0} (1 + |\eta|(2m+1))^k \left\| \frac{f_m(\eta)}{|\eta|^{\frac{1}{4}}} \right\|_{L^2}^2 \,.$$

Cost of $-\Delta$ is "$(2m+1)|\eta|$".

The randomisation

We further decompose dyadically in $\eta \in [I, 2I]$, $I \in 2^{\mathbb{Z}}$.

$$u = \sum_{\substack{m \geqslant 0 \\ I \in 2^{\mathbb{Z}}}} u_{m,I} = \sum_{A \in 2^{\mathbb{N}}} \sum_{\substack{m,I \\ 1+(2m+1)I \in [A,2A]}} u_{m,I} = \sum_{A \in 2^{\mathbb{N}}} u_A \,,$$

with $\mathcal{F}_{y \to \eta} u_{m,I} = \mathbf{1}_{\eta \in [I,2I]} f_m(\eta) h_m(|\eta|^{\frac{1}{2}} x)$. Observe that:

$$\|u_A\|_{H^s} \sim A^{\frac{s}{2}} \|u_A\|_{L^2} \,.$$

Definition

For $u_0 = \sum_{m \geqslant 0, I \in 2^{\mathbb{Z}}} u_{m,I}$ we introduce a randomisation

$$u_0^\omega = \sum_{\substack{m \geqslant 0 \\ I \in 2^{\mathbb{Z}}}} g_{m,I}^\omega u_{m,I} \text{ and } \mu_{u_0} = \text{Im Measure of } \mathbb{P} \,,$$

where $(g_{m,I})_{m,I}$ are i.i.d. Gaussian random variables.

Adapted space

The space $\mathcal{X}^k_\rho$ is defined by the norm

$$\|u\|^2_{\mathcal{X}^k_\rho} = \sum_{\substack{m\geqslant 0\\ l\in 2^{\mathbb{Z}}}} (1+(2m+1)l)^k \langle l\rangle^\rho \|u_{m,l}\|^2_{L^2} \, ,$$

is a space of H^k_G functions, with additional $\frac{\rho}{2}$ regularity in the variable y.

Lemma

For any $\varepsilon > 0$, $\mathcal{X}^k_\rho \not\subset H^{k+\varepsilon}_G$.

Proposition (Properties of the measures)

$\cup_{u_0\in\mathcal{X}^k_1} \operatorname{supp}(\mu_{u_0}) \subset H^k_G \setminus \bigcup_{\varepsilon>0} H^{k+\varepsilon}_G$ *is dense in* $H^k_G \setminus \bigcup_{\varepsilon>0} H^{k+\varepsilon}_G$.

Why these anisotropic spaces?

Overcoming the obstruction $u_L \notin H_G^{\frac{3}{2}^+}$

Heuristics suggests:

$$\langle \nabla \rangle^{\frac{3}{2}^+} (|u_L^\omega|^2 u_L^\omega) \simeq |u_L^\omega|^2 \langle \nabla \rangle^{\frac{3}{2}^+} u_L^\omega$$

With "Good measure construction" $u_L^\omega \notin H_G^{\frac{3}{2}^+}$.

Next attempt by Hölder:

$$\| |u_L^\omega|^2 u_L^\omega \|_{H_G^{\frac{3}{2}^+}} \lesssim \| u_L^\omega \|_{L^8}^2 \| u_L^\omega \|_{W_G^{\frac{3}{2}^+,4}} .$$

Can randomness can prove $u_L^\omega \in W_G^{\frac{3}{2}^+,4}$?

Probabilistic toolbox: part I

Lemma (Decoupling estimate)

For all $r \geqslant 2$ and all complex numbers $(a_n)_{n\geqslant 0}$ there holds:

$$\Big\|\sum_{n\geqslant 0} a_n g_n^\omega\Big\|_{L^r_\Omega} \lesssim \sqrt{r}\left(\sum_{n\geqslant 0}|a_n|^2\right)^{\frac{1}{2}} \quad (g_n^\omega \text{ are i.i.d. Gaussians}).$$

- Implies that outside a set of probability $\leqslant e^{-cR^2}$,

$$\Big|\sum_{n\geqslant 0} a_n g_n^\omega\Big| \leqslant R\left(\sum_{n\geqslant 0}|a_n|^2\right)^{\frac{1}{2}}.$$

- Deterministic estimate:

$$\Big|\sum_{n=N}^{2N} a_n g_n^\omega\Big| \leqslant \sqrt{N}\left(\sum_{n=N}^{2N}|a_n g_n^\omega|^2\right)^{\frac{1}{2}}.$$

Randomness + deterministic input

$$\langle\nabla\rangle^{\frac{3}{2}} u_L^\omega(t,x) = \sum_{m,l} g_{m,l}^\omega \langle\nabla\rangle^{\frac{3}{2}}(e^{it\Delta}u_{m,l}),$$

$$\left\|\langle\nabla\rangle^{\frac{3}{2}} u_L^\omega(t,x)\right\|_{L^r_\Omega} \lesssim \sqrt{r}\left(\sum_{m,l}|\langle\nabla\rangle^{\frac{3}{2}}(e^{it\Delta}u_{m,l}(x))|^2\right)^{\frac{1}{2}}.$$

Fourier localisation: $\|\langle\nabla\rangle^{\frac{3}{2}} u_{m,l}\|_{L^p}^2 \lesssim (1+(2m+1)l)^{\frac{3}{2}}\|u_{m,l}\|_{L^p}^2$.

$$\|u_L^\omega\|_{L^r_\Omega L^q_T W^{3/2,4}_G} \lesssim \sqrt{r}\,T^{\frac{1}{q}}\left(\sum_{m,l}(1+(2m+1)l)^{\frac{3}{2}}\|u_{m,l}\|_{L^4}^2\right)^{\frac{1}{2}}$$

Probability decouples L^4 norms!

Lemma (Deterministic input)

$$\|u_{m,l}\|_{L^4} \lesssim (1+(2m+1)l)^{-\frac{1}{8}}\langle l\rangle^{\frac{1}{2}}\|u_{m,l}\|_{L^2}.$$

Main linear random estimate

Proposition (Regularity gain in L^4 norm)

Let $q \geqslant 1$. Outside a set of probability at most e^{-cR^2} there holds

$$\|u_L^\omega\|_{L^q_T W_G^{\frac{3}{2}^+,4}} \leqslant T^{\frac{1}{q}} R \|u_0\|_{\mathcal{X}_1^{\frac{5}{4}^+}}$$

Remarks: in L^4 norms we have an $1/4$ derivative gain. In L^2 norm this gain is 0, no smoothing in usual Sobolev spaces.

This is the maximum gain: generally on $W_G^{s,p}$ the gain is computable by the same method.

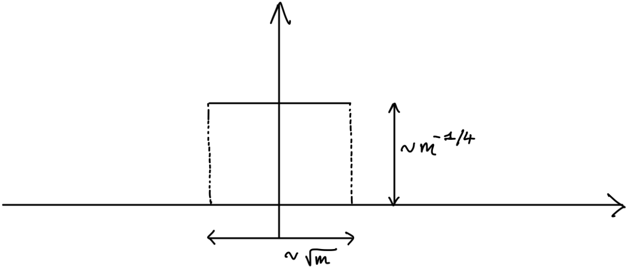
The proof of the lemma

$$\|u_{m,I}\|_{L^4} \lesssim \|\hat{u}_{m,I}\|_{L^{4/3}_\eta L^4_x} = \|f_m(\eta)\mathbf{1}_{\eta\sim I}\|h_m(|\eta|^{1/2}x)\|_{L^4_x}\|_{L^{4/3}_\eta}$$

Lemma

(Linear Hermite estimate) $\|h_m\|_{L^4} \lesssim m^{-1/8}$.

Because there is only a gain in m this explains the $\mathcal{X}^{k,\rho}$ spaces.



Local theory: probabilistic improvement

Writing $\langle\nabla\rangle^{\frac{3}{2}^+}(|u_L^\omega|^2 u_L^\omega) \simeq |u_L^\omega|^2 \langle\nabla\rangle^{\frac{3}{2}^+} u_L^\omega$ by Hölder:

$$\left\| |u_L^\omega|^2 u_L^\omega \right\|_{L_T^1 H_G^{\frac{3}{2}^+}} \lesssim \|u_L^\omega\|_{L_T^4 L_x^8}^2 \|u_L^\omega\|_{L_T^2 W_G^{\frac{3}{2}^+,4}}$$
$$\lesssim T^{\frac{1}{2}} \|u_L^\omega\|_{L_T^4 L_x^8}^2 \|u_0\|_{\mathcal{X}_1^{5/4}} .$$

Now only requires regularity $5/4$ on u_0 and not $3/2$!

Anisotropic space $\mathcal{X}_1^k$: Hermite gains powers of m, not $A \sim mI$.

To go further: multilinear theory.

1. The probabilistic local well-posedness framework ✔
2. The randomisation ✔
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Multilinear estimates

Theorem (Key multilinear estimates)

1. *Outside a set of probability at most* e^{-cR^2} *there holds*

$$\||u_L^\omega|^2 u_L^\omega\|_{L^q_T H_G^{\frac{3}{2}+}} \lesssim R^3 T^{\frac{1}{q}} \|u_0\|^3_{\mathcal{X}_1^1}.$$

2. *Outside a set of probability at most* e^{-cR^2} *the following is true. For every* $v, w \in \mathcal{C}^0([0,T], H_G^{\frac{3}{2}^+})$,

$$\|u_L^\omega v w\|_{L^q_T H_G^{\frac{3}{2}+}} \lesssim R^3 T^{\frac{1}{q}} \|u_0\|_{\mathcal{X}_1^1} \|v\|_{H^{\frac{3}{2}+}} \|w\|_{H_G^{\frac{3}{2}+}}.$$

We concentrate on explaining (1).

Such estimates give a local theory in $H_G^{1^+}$

Probabilistic reductions

Time averages is not used: replace $u_L(t) = e^{it\Delta} u_0^\omega$ by u_0^ω.

Proposition (Random-Random bilinear estimate)

Outside a set of probability at most e^{-cR^2} there holds

$$\|(u_0^\omega)^2\|_{H_G^{k+\frac{1}{2}}} \leqslant R^2 \|u_0\|_{\mathcal{X}_1^k}^2 .$$

$$(u_0^\omega)^2 = \sum_{B \gg A} (u_0^\omega)_A (u_0^\omega)_B + \underbrace{\sum_{A \sim B} (u_0^\omega)_A (u_0^\omega)_B}_{\text{Split derivatives}}$$

Use of randomness

$$\langle\nabla\rangle^{k+\frac{1}{2}}(u_0^\omega)^2 = \sum_{mI\sim A\ll B\sim nJ} \langle\nabla\rangle^{k+1/2}(u_{m,I}u_{n,J})g_{m,I}^\omega g_{n,J}^\omega \,.$$

Theorem (Order 2 Wiener Chaos)

Let $\mathcal{I}$ be countable and $(g_n)_{n\in\mathcal{I}}$ complex i.i.d. standard Gaussians. Then for all $r\geqslant 2$,

$$\Big\| \sum_{n,n'\in\mathcal{I}} \Psi_{n,n'} g_n^\omega g_{n'}^\omega \Big\|_{L^r(\Omega)} \lesssim r \Big\| \sum_{n,n'\in\mathcal{I}} \Psi_{n,n'} g_n^\omega g_{n'}^\omega \Big\|_{L^2(\Omega)} \,.$$

Minkowski $r\geqslant 2$:

$$\|(u_0^\omega)^2\|^2_{L^r_\omega H_G^{k+\frac{1}{2}}} \leqslant r^2 \Big\| \sum_{mI\sim A\ll B\sim nJ} \|u_{m,I}u_{n,J}\|_{H_G^{k+\frac{1}{2}}} g_n^\omega g_m^\omega \Big\|^2_{L^2_\omega} \,.$$

Reduction to deterministic estimates

Expand L^2_ω norm. Use: $\mathbb{E}[g^\omega_{n_1} g^\omega_{m_1} \bar{g}^\omega_{n_2} \bar{g}^\omega_{m_2}] = 1$ iff $\{n_1, m_1\} = \{n_2, m_2\}$.

$$\|(u_0^\omega)^2\|^2_{L^r_\omega H_G^{k+\frac{1}{2}}} \leqslant r^2 \sum_{mI \sim A \ll B \sim nJ} \|u_{m,I} u_{n,J}\|^2_{H_G^{k+\frac{1}{2}}}$$

Proposition (Paradifferential input)

Let $mI \sim A$ and $nJ \sim B$,

$$\|u_{m,I} v_{n,J}\|^2_{H_G^{k+\frac{1}{2}}} \lesssim \max\{A, B\}^{k+1/2} \|u_{m,I} v_{n,J}\|^2_{L^2}.$$

$$\|(u_0^\omega)^2\|^2_{L^r_\omega H_G^{k+\frac{1}{2}}} \leqslant r^2 B^{k+\frac{1}{2}} \sum_{mI \sim A \ll B \sim nJ} \|u_{m,I} u_{n,J}\|^2_{L^2}$$

The main deterministic bilinear estimate

Proposition (Key bilinear estimate)

For any n, m, I, J there holds:

$$\|u_{m,I}v_{n,J}\|_{L^2}^2 \lesssim \min\left\{\frac{J\langle I\rangle}{A^{\frac{1}{2}}}, \frac{I\langle J\rangle}{B^{\frac{1}{2}}}\right\}\|u_{m,I}\|_{L^2}^2\|v_{n,J}\|_{L^2}^2 .$$

We write:

$$\begin{aligned}\mathcal{F}_{y\to\eta}(u_{m,I}v_{n,J})(x,\eta) &= \hat{u}_{m,I} * \hat{v}_{n,J}(x,\eta)\\ &= \int_{\eta_1+\eta_2=\eta} f_m(\eta_1)f_n(\eta_2)\mathbf{1}_{(|\eta_1|,|\eta_2|)\in[I,2I]\times[J,2J]}h_m(|\eta_1|^{\frac{1}{2}}x)h_n(|\eta_2|^{\frac{1}{2}}x)\end{aligned}$$

$\rightsquigarrow$ Need to study $h_n h_m$

Bilinear estimates for Hermite functions

Lemma

For any n, m, η_1, η_2,

$$\|h_m(|\eta_1|^{\frac{1}{2}}\cdot)h_n(|\eta_2|^{\frac{1}{2}}\cdot)\|_{L^2}^2 \lesssim \min\left\{\frac{1}{\sqrt{|\eta_1|(2n+1)}}, \frac{1}{\sqrt{|\eta_2|(2m+1)}}\right\}.$$

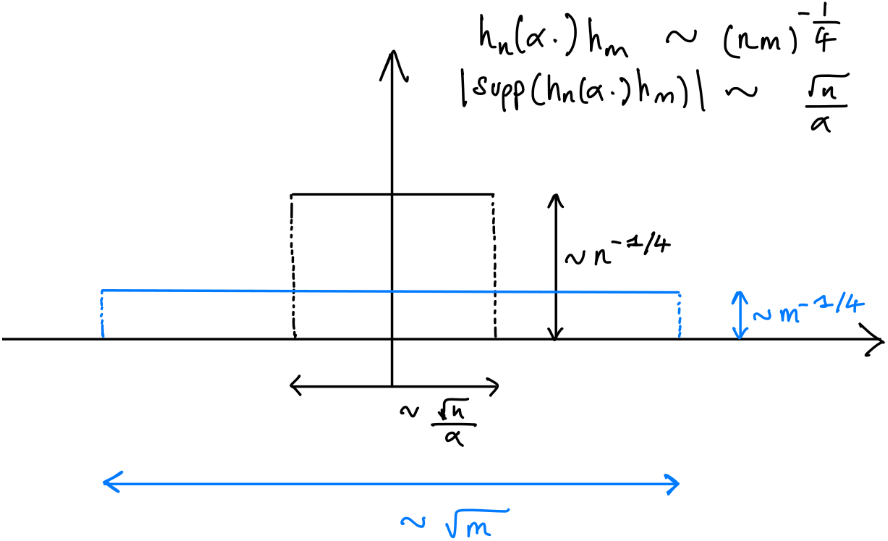
After rescaling, reduces to:

$$\|h_m h_n(\alpha\cdot)\|_{L^2}^2 \lesssim \frac{1}{\alpha\sqrt{2m+1}},$$

for $(2n+1) \ll \alpha^2(2m+1)$.

This is the bilinear Hermite input.

Bilinear Hermite estimate



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Mixed terms

Proposition (content of Section 7 in the paper)

Outside a set of probability e^{-cR^2} there holds:

$$\forall v \in H^k_G,\ \|u^\omega v\|_{H_G^{k+\frac{1}{2}}} \leqslant R^2 \|u_0\|_{X_1^k} \|v\|_{H_G^{k+\frac{1}{2}}}\,.$$

We write:

$$\hat{u}_A^\omega = \sum_{m,I} f_{m,I}^\omega(\eta) h_{m,I}(|\eta|^{\frac{1}{2}} x) \quad \hat{v}_B = \sum_{n,J} g_{n,J}(\eta) h_{n,J}(|\eta|^{\frac{1}{2}} x)$$

Write:

$$\|u_A^\omega v_B\|_{L^2}^2 = \int_x \hat{u}_A^\omega * \bar{\hat{u}}_A^\omega \hat{v}_B * \bar{\hat{v}}_B$$

Mixed terms (end)

Re-write everything as

$$\|u_A^\omega v_B\|_{L^2}^2 = \sum_{\psi=(n_1,n_2,m_1,m_2,\ldots)} \mathbf{J}_\psi^\omega \mathbf{K}_\psi$$

Hölder estimates and deterministic treatment available on $\mathbf{K}$: just as before.

Khinchine only possible on $\mathbf{J}^\omega$: Important! Khinchine set of probability depends on the coefficients to which it is applied!

Questions?