New results on Lieb-Thirring inequalities

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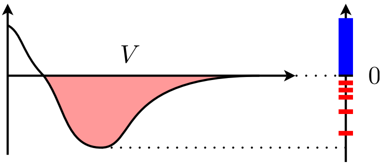
Lieb-Thirring inequality



$|\lambda_1(-\Delta + V)| \leq \|V_-\|_{L^\infty}$

Theorem (Lieb-Thirring '75–76, Cwikel-Lieb-Rozenblum '72–77, Weidl '96)

For all $V \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d, \mathbb{R})$, we have

$$\sum_{n=1}^{\infty} |\lambda_n(-\Delta + V)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\, dx, \qquad \gamma \begin{cases} \geq \frac{1}{2} & \text{for } d = 1 \\ > 0 & \text{for } d = 2 \\ \geq 0 & \text{for } d \geq 3 \end{cases}$$

where $\lambda_n(-\Delta + V)$ is the nth min-max level of the Friedrichs extension of $-\Delta + V$, which is the nth eigenvalue when it exists and 0 otherwise.

- $\gamma = 1$: kinetic energy of fermions, stability of matter
- $\gamma = 3/2$, $d = 1$: integrable, KdV solitons, inverse scattering
- $\gamma = 0$: conformal invariance, Yamabe problem
- $\gamma = 2$, $d = 2$: dimension of attractors for Navier-Stockes, magnetohydrodynamics

R.L. Frank, The Lieb-Thirring inequalities: Recent results and open problems, arXiv:2007.09326 (2020)

Motivation: stability of matter ($\gamma = 1$)

- Two kinds of elementary particles in nature: **fermions** and **bosons**
  - **Dyson (1967):** bosonic matter unstable (collapse due to Coulomb forces)
  - **Dyson-Lenard (1967–68):** fermionic matter stable, involved analytical proof
  - **Lieb-Thirring (1975):** much simpler proof of stability based on LT at $\gamma = 1$

- N quantum particles described by wavefunction $\Psi(x_1, ..., x_N) \in L^2((\mathbb{R}^3)^N, \mathbb{C})$ which is either symmetric (bosons) or antisymmetric (fermions)

$$\text{lowest eigenvalue of } \sum_{j=1}^{N} \left(-\Delta_{x_j} + V(x_j) \right) = \begin{cases} N\lambda_1(-\Delta + V) & \text{sym.} \\ \sum_{j=1}^{N} \lambda_j(-\Delta + V) & \text{antisym.} \end{cases}$$

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This talk: best constant $L_{\gamma,d}$

- important for applications, in particular at $\gamma = 1$
- several possible cases depending on γ
- statistical mechanics interpretation: phase diagram with $\gamma \gg 1 \sim$ large entropy

Lieb-Thirring is stronger than Gagliardo-Nirenberg-Sobolev

$$\left|\lambda_1(-\Delta+V)\right|^\gamma \leq L^{(1)}_{\gamma,d}\int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\,dx$$

is **equivalent** to the Gagliardo-Nirenberg-Sobolev inequality $H^1(\mathbb{R}^d)\hookrightarrow L^p(\mathbb{R}^d)$

$$\left(\int_{\mathbb{R}^d}|u(x)|^p\,dx\right)^{\frac{4}{d(p-2)}} \leq C^{\rm GNS}_{p,d}\left(\int_{\mathbb{R}^d}|u(x)|^2\,dx\right)^{\frac{(2-d)p+2d}{d(p-2)}}\int_{\mathbb{R}^d}|\nabla u(x)|^2\,dx$$

and the best constants are related by $L^{(1)}_{\gamma,d}=\left(\frac{2\gamma}{2\gamma+d}\right)^{\gamma+\frac{d}{2}}\left(\frac{d}{2\gamma}\right)^{\frac{d}{2}}\left(C^{\rm GN}_{p,d}\right)^{\frac{d}{2}}$

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Idea: $V\in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d,\mathbb{R})$ variable dual to $|u|^2\in L^{\frac{p}{2}}(\mathbb{R}^d,\mathbb{R})$, with $\gamma+\frac{d}{2}=\left(\frac{p}{2}\right)'$

$$2\leq p\begin{cases}\leq\infty & \text{for } d=1\\ <\infty & \text{for } d=2\\ \leq\frac{2d}{d-2} & \text{for } d\geq 3\end{cases}\qquad\Longleftrightarrow\qquad \gamma\begin{cases}\geq\frac{1}{2} & \text{for } d=1\\ >0 & \text{for } d=2\\ \geq 0 & \text{for } d\geq 3\end{cases}$$

Proof. Variational principle $\lambda_1(-\Delta+V)=\inf\left\{\int_{\mathbb{R}^d}|\nabla u|^2+V|u|^2\ :\ \int_{\mathbb{R}^d}|u|^2=1\right\}$

$$\int_{\mathbb{R}^d}|\nabla u|^2+V|u|^2\geq\|\nabla u\|^2_{L^2}-\|V_-\|_{L^{\gamma+\frac{d}{2}}}\|u\|^2_{L^p}\geq\|\nabla u\|^2_{L^2}-\|V_-\|_{L^{\gamma+\frac{d}{2}}}\left(C^{\rm GN}_{p,d}\|\nabla u\|^2_{L^2}\right)^{\frac{d(p-2)}{2p}}$$

Weyl's law and the semi-classical constant

- We can take a constant potential in LT!

Theorem (Weyl 1911)

Let Ω be any bounded open set with $|\partial\Omega| = 0$. Then, for every $\mu > 0$ we have

$$\sum_{n\geq 1} |\lambda_n(-\Delta - \mu\mathbb{1}_{\ell\Omega})|^\gamma \underset{\ell\to\infty}{\sim} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \left(|p|^2 - \mu\mathbb{1}_{\ell\Omega}\right)_-^\gamma \frac{dx\,dp}{(2\pi)^d} = L^{sc}_{\gamma,d} \underbrace{\mu^{\gamma+\frac{d}{2}}\ell^d|\Omega|}_{\int_{\mathbb{R}^d}(-\mu\mathbb{1}_{\ell\Omega})_-^{\gamma+d/2}}$$

with $L^{sc}_{\gamma,d} = \int_{\mathbb{R}^d} \left(|p|^2 - 1\right)_-^\gamma \frac{dp}{(2\pi)^d}$.

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- We have $L_{\gamma,d} \geq \max\left(L^{(1)}_{\gamma,d}, L^{sc}_{\gamma,d}\right)$. Could there be equality?

Theorem (Optimality of constant potentials)

For all $d \geq 1$, we have

$$L_{\gamma,d} \begin{cases} = L^{sc}_{\gamma,d} & \text{for } \gamma \geq 3/2, \\ > L^{sc}_{\gamma,d} & \text{for } \gamma < 1. \end{cases}$$

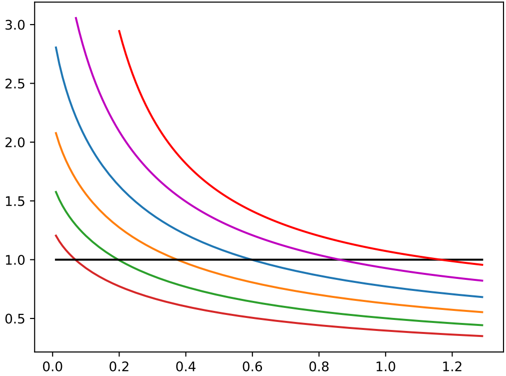
Lieb-Thirring '76, Aizenman-Lieb '78, Laptev-Weidl '00, Helffer-Robert '90

Comparing $L^{(1)}_{\gamma,d}$ and $L^{\rm sc}_{\gamma,d}$

Theorem

In all dimensions $d \geq 1$, $\gamma \mapsto L^{(1)}_{\gamma,d}/L^{sc}_{\gamma,d}$ is strictly decreasing. For $d \leq 7$, it is crossing 1 at a unique point, which decreases with the dimension and equals $3/2$ in $d = 1$. For $d \geq 8$, we have $L^{(1)}_{\gamma,d} < L^{sc}_{\gamma,d}$ for all $\gamma \geq 0$.

Glaser-Grosse-Martin '78, Frank-Gontier-ML '21



The case of N eigenvalues

$$L^{(N)}_{\gamma,d} := \sup_{\substack{V\in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)\\ V_-\neq 0}} \frac{\sum_{n=1}^N |\lambda_n(-\Delta+V)|^\gamma}{\int_{\mathbb{R}^d} V(x)_-^{\gamma+\frac{d}{2}}\,dx} = \sup_{\substack{0\geq V\in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)\\ \int_{\mathbb{R}^d}|V|^{\gamma+\frac{d}{2}}=1}} \sum_{n=1}^N |\lambda_n(-\Delta+V)|^\gamma \underset{N\to\infty}{\nearrow} L_{\gamma,d}$$

Theorem (Existence, Frank-Gontier-ML '21)

Let $\gamma > \frac{1}{2}$ in $d = 1$, $\gamma > 0$ in $d \geq 2$.

- $L^{(N)}_{\gamma,d}$ *always* **has optimizers**. *Those may have $M < N$ negative eigenvalues, but are then also optimizers for $L^{(M)}_{\gamma,d} = L^{(N)}_{\gamma,d}$. Assuming N negative eigenvalues, then the eigenfns u_j of $-\Delta + V$ solve*

$$\Bigg(-\Delta \underbrace{-\left(C\sum_{j=1}^N |\lambda_j|^{\gamma-1}|u_j|^2\right)^{\frac{1}{\gamma+\frac{d}{2}-1}}}_{=V}\Bigg)u_j = \lambda_j\, u_j,$$

where $C = \frac{2\gamma}{(d+2\gamma)L^{(N)}_{\gamma,d}}$. We have $|V(x)| \leq C\exp\left(-\frac{2\sqrt{\lambda_N}}{\gamma+\frac{d}{2}-1}|x|\right)$.

• More precisely, any maximizing sequence (V_n) normalized in $L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$ can be decomposed into **bubbles**:

$$\lim_{n\to\infty}\left\|V_n-\sum_{k=1}^K V^{(k)}(\cdot - x_n^{(k)})\right\|_{L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)}=0$$

up to a subsequence, with $V^{(k)}$ optimizers for $L_{\gamma,d}^{(N_k)}=L_{\gamma,d}^{(N)}$, $\sum_{k=1}^K N_k=N$, and $|x_n^{(k)}-x_n^{(k')}|\to\infty$ for $k\neq k'$.

• At $\gamma=0$ in $d\geq 3$, similar result but need to count zero-energy resonances in addition to the negative eigenvalues. Dilations appear in bubble decomposition. Maximizers solve

$$\left(-\Delta-\left(C\sum_{j=1}^N|f_j|^2\right)^{\frac{2}{d-2}}\right)f_j=0$$

with $f_j\in\dot{H}^1(\mathbb{R}^d)$ (zero-energy resonances). We have $V(x)\sim c/|x|^4$ at infinity.

Remarks:

- Concentration-compactness
- Difficulty: $V\in L^{\gamma+d/2}(\mathbb{R}^d)\mapsto$ eigenvalues/functions is highly nonlinear
- For $\gamma>0$, subcriticality seen for eigenfunctions. Problem is **locally compact** [if $V_n\rightharpoonup V$ weakly but tightly in $L^{\gamma+d/2}(\mathbb{R}^d)$, then spectrum converges]
- At $\gamma=0$, optimizers **only have zero-energy resonances**. Those are true eigenfunctions in dimensions $d\geq 5$

Binding through (nonlinear) quantum tunnelling for $\gamma > 2 - d/2$

Theorem (Binding for $\gamma > \max(0, 2 - d/2)$, Frank-Gontier-ML '21)

Let $\gamma > \max\left(0, 2 - \frac{d}{2}\right)$. For every $N \geq 1$, we have

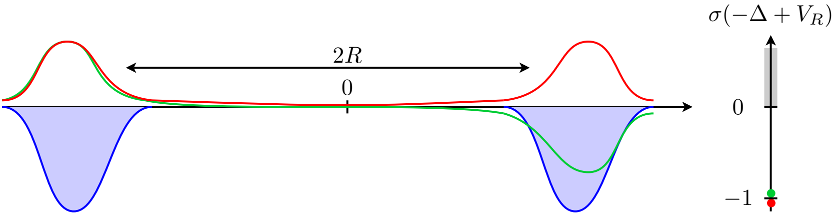
$$L^{(2N)}_{\gamma,d} > L^{(N)}_{\gamma,d}$$

hence $L_{\gamma,d}$ cannot be given by a potential having finitely many bound states:

$$L_{\gamma,d} > L^{(N)}_{\gamma,d}, \qquad \text{for all } N \geq 1.$$

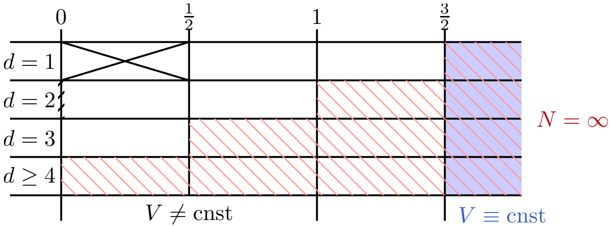
At $\gamma = 0$ we have $L^{(2)}_{0,d} = L^{(1)}_{0,d}$ for all $d \geq 3$.

Proof: Use $V^{(2)}_R(x) = -\left(-V^{(1)}(x - Re_1)^{\gamma+\frac{d}{2}-1} - V^{(1)}(x + Re_1)^{\gamma+\frac{d}{2}-1} \right)^{\frac{1}{\gamma+\frac{d}{2}-1}}$



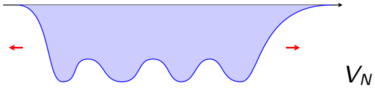
There is an attraction if and only if $\gamma > \max(0, 2 - d/2)$

Summary and a conjecture

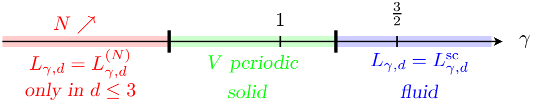


Conjecture (Frank-Gontier-ML '21)

For $\gamma > \max\{0, 2 - d/2\}$, Lieb-Thirring "optimizer" exists and is a **periodic function** (possibly constant).

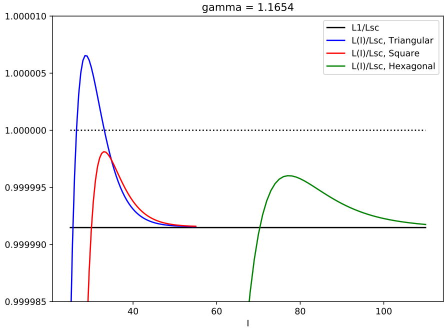


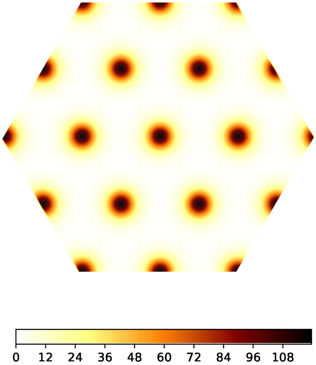
Speculative "phase diagram":



Numerics in 2D

In $d = 2$ we found **periodic potentials which beat both** $L^{(1)}_{\gamma,2}$ and $L^{\mathrm{sc}}_{\gamma,2}$





Important technical difficulties:

- very small difference between the lattices and the fluid
- binding energy really seems exponentially small, hard to catch
- need very high precision and the problem is nonlinear

Lieb-Thirring and KdV at $\gamma = \frac{3}{2}$ in 1D

- **Scattering problem in 1D:**

$$\begin{cases} \left(-\frac{d^2}{dx^2} + V\right)\psi_k = k^2\psi_k \\ \psi_k(x) \underset{x\to+\infty}{\sim} e^{ikx} \end{cases} \qquad \Longrightarrow \qquad \psi_k(x) \underset{x\to-\infty}{\sim} a(k)e^{ikx} + b(k)e^{-ikx}$$

with $|a(k)|^2 = 1 + |b(k)|^2$

Theorem (Trace formula, Zaharov-Fadeev '76)

For a sufficiently nice V, we have

$$\sum_j |\lambda_j(V)|^{\frac{3}{2}} + \frac{3}{\pi}\int_{\mathbb{R}} \log|a(k)|\, k^2 dk = \frac{3}{16}\int_{\mathbb{R}} V(x)^2\, dx$$

Corollary

At $\gamma = 3/2$ in $d = 1$, we have

$$L_{\frac{3}{2},1} = L^{(N)}_{\frac{3}{2},1} = L^{sc}_{\frac{3}{2},1} = \frac{3}{16}, \qquad \forall N \in \mathbb{N}.$$

Equality for relectionless potentials ($|a| \equiv 1$), which are exactly the KdV N solitons.

Gardner-Greene-Kruskal-Miura '74, Deift-Trubowitz '79, Frank-Gontier-ML '21

The $N=1$ KdV soliton is $V(x)=-2/\cosh^2(x)$ and the N-solitons are

$$V_{\vec{X}}(x)=-2\frac{d^2}{dx^2}\log\det\left(\mathbb{1}_N+\frac{e^{-|\lambda_j|^{\frac12}(x-X_j)-|\lambda_k|^{\frac12}(x-X_k))}}{|\lambda_j|^{\frac12}+|\lambda_k|^{\frac12}}\right),\qquad \sum_{j=1}^N|\lambda_j|^{\frac32}=\frac{3}{16}$$

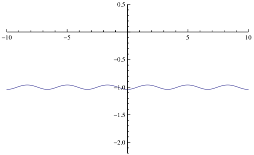
then $V_{\vec{X}(t)}(x)$ solves $\partial_t V+\partial_x^3 V-6V\partial_x V=0$ with $X_j(t)=X_j+4t\lambda_j$

Theorem (Periodic optimizers in 1D, Frank-Gontier-ML '21)

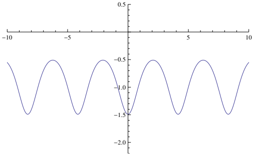
Let $\gamma=\frac32$ and $d=1$. For all $0<k<1$, the $\ell=2K(k)$ periodic Lamé potential $V(x)=2k^2\mathrm{sn}(x|k)^2-1-k^2$ is also an optimizer of $L_{\frac32,1}=\frac{3}{16}$, in the sense that

$$\lim_{R\to\infty}\frac{\sum_{n=1}^\infty|\lambda_n(-\Delta-V\mathbb{1}_{B_R})|^{\frac32}}{\int_{B_R}V^2}=\frac{3}{16}.$$

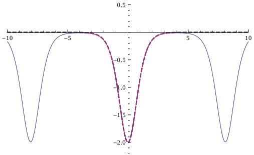
Here sn and K are the Jacobi elliptic function and complete elliptic integral of the first kind, with modulus k.



$k=0.2$ ($\ell=3.32$)



$k=0.7$ ($\ell=4.15$)



$k=0.995$ ($\ell=8.08$)

Conclusion

- **Lieb-Thirring inequality**
  - is important for applications in mathematical physics
  - has links with many areas of analysis

- **Best constant**
  - still unknown in many important cases
  - should be interpreted in the framework of statistical mechanics, with γ playing a role similar to a temperature
  - all the phases occur at $\gamma = 3/2$ in 1D, which is an integrable system
  - more complicated phase diagram expected in $d \geq 2$