Full description of Benjamin-Feir instability of Stokes wave in deep water

A. Maspero

joint work with M. Berti and P. Ventura

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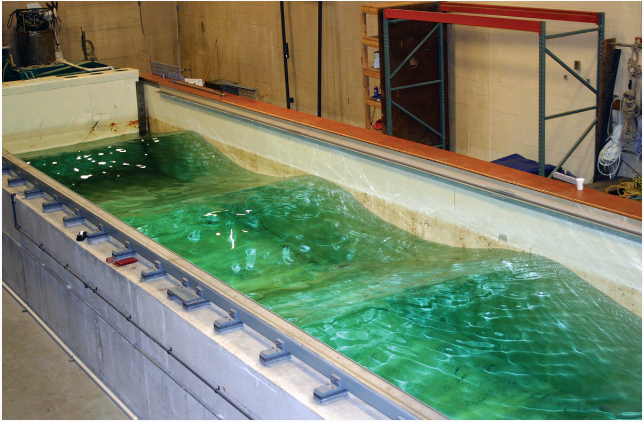
CY Cergy Paris Université “CY Days in Nonlinear Analysis ”

SISSA

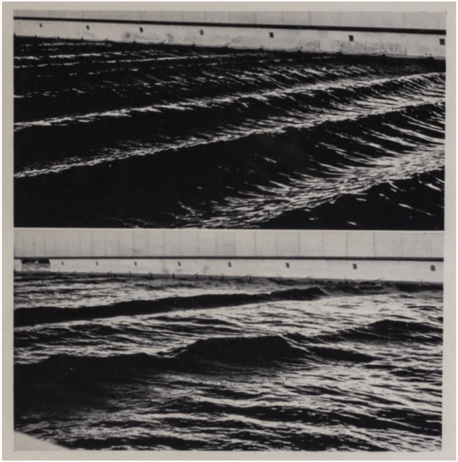
1967

Stokes waves are traveling 1 dimensional, 2π-periodic solutions of water waves



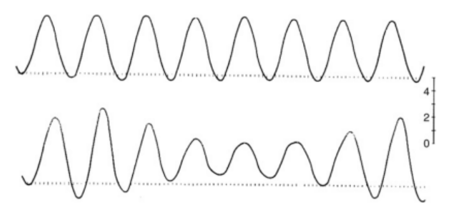
It was widely believed in the 60s that Stokes waves are **stable** solutions

Benjamin-Feir's experiments to prove the stability of Stokes waves keep failing



''Deep-water wave trains were generated at one end of a long tank and were observed travelling many wavelengths. It was found that such a wavetrain may develop conspicuous irregularities if it travels far enough, even when departures from periodicity can hardly be detected near the origin. And eventually, at great distances from the origin, the train may become completely disintegrated and its energy redistributed over a broad spectrum''

Benjamin: *Instability of periodic wavetrains in nonlinear dispersive systems*, 1967



Benjamin-Feir (or moduational) instability

BF heuristic physical mechanism: "with long wave perturbations, Stokes wave becomes unstable"

- Many experimental and numerical results, possible mechanism for formation of rogue waves

- Rigorous mathematical results for Water Waves:
  1. Bridges-Mielke '95 (finite depth), linear instability
  2. Nguyen-Strauss 2020 (infinite depth), linear instability
  3. Hur-Yang 2020 (finite depth), linear instability (different proof)
  4. Chen-Su 2020 (infinite depth), Nonlinear instability
  5. Rousset-Tzvetkov 2011: linear and nonlinear transversal instability for solitary water-waves

- Many results for dispersive PDEs (NLS, gKdV, Whitham, ...) by
  Segur-Henderson-Carter-Hammack, Gallay-Haragus, Haragus-Kapitula, Bronski-Johnson, Johnson, Hur-Johnson, Bronski-Hur-Johnson, Hur-Pandey, Leisman-Bronski-Johnson-Marangell, Jin-Liao-Lin

"Take-home theorem": Berti, M., Ventura 2021

Full description of the **spectrum** near zero of the linearized water waves at **small** amplitude Stokes waves acting on **long wave** periodic perturbations

Mathematics of Benjamin-Feir instability

Water Waves in 2D: Euler equations for an incompressible, irrotational fluid in deep water $\mathcal{D}_\eta(t) = \{y < \eta(t,x)\}$ under gravity.

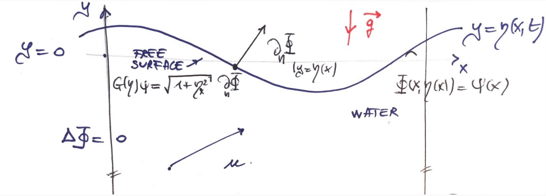
Unknown: $\eta(t,x)$ wave profile

$\psi(t,x) = \Phi(t,x,\eta(t,x))$ trace of velocity potential at the border

Zakharov formulation of WW

$$\begin{cases} \eta_t = G(\eta)\psi \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} \end{cases}$$

$$\begin{cases} -\Delta\Phi = 0 & \text{in } -h < y < \eta(t,x) \\ \Phi = \psi & \text{at } y = \eta(t,x) \\ \partial_y\Phi = 0 & \text{at } y = -\infty \end{cases}$$



Dirichlet–Neumann operator: $G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n\Phi|_{y=\eta(x)} = (\Phi_y - \eta_x\Phi_x)|_{y=\eta(x)}$

- $G(\eta)$ selfadjoint, $\eta \mapsto G(\eta)$ analytic from $H^s \to \mathcal{L}(H^s, H^{s-1})$
- $G(0)\psi = |D|\psi$

Hamiltonian structure and symmetries

WW is a Hamiltonian system with $\eta(x)$ and $\psi(x)$ as canonical Darboux coordinates

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = J \begin{pmatrix} \nabla_\eta^{L^2} H(\eta,\psi) \\ \nabla_\psi^{L^2} H(\eta,\psi) \end{pmatrix}, \quad J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$$

Hamiltonian expressed in terms of (η, ψ)

$$H(\eta,\psi) = \frac{1}{2}\int_{\mathbb{T}} \psi(x)\, G(\eta)\psi(x)\, dx + \frac{1}{2}\int_{\mathbb{T}} g\eta^2\, dx$$

Reversibility

$$H \circ \rho = H, \qquad \rho(\eta(x),\psi(x)) := (\eta(-x), -\psi(-x))$$

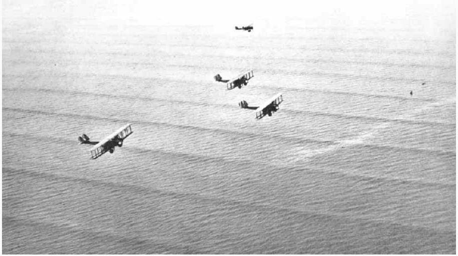
Space invariance

$$H \circ \tau_\theta = H, \quad (\tau_\theta u)(x) := u(x+\theta)$$

Periodic traveling waves solution of WW

$$\eta(t,x) = \breve{\eta}(x-ct),$$
$$\psi(t,x) = \breve{\psi}(x-ct)$$

2π-periodic *profiles* $\breve{\eta}(x), \breve{\psi}(x)$, *speed* $c \in \mathbb{R}$



In a reference frame in translational motion with constant speed c, the WW equations are

$$\begin{cases} \eta_t = c\eta_x + G(\eta)\psi \\ \psi_t = c\psi_x - g\eta - \frac{\psi_x^2}{2} + \frac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 \end{cases}$$

Stokes waves = equilibrium steady solutions

Theorem (Stokes, Levi-Civita, Struik, Nekrasov ....)

There exist $\epsilon_0 > 0$ *and analytic solutions* $(\eta_\epsilon(x), \psi_\epsilon(x), c_\epsilon)$, *parameterized by the amplitude* $|\epsilon| \leq \epsilon_0$ *with*

- $\eta_\epsilon(x)$, $\psi_\epsilon(x)$ 2π *periodic in* x
- $\eta_\epsilon(x)$ *even,* $\psi_\epsilon(x)$ *is odd*
- *expand as*

$$
\begin{aligned}
\eta_\epsilon(x) &= \epsilon\cos(x) + \frac{\epsilon^2}{2}\cos(2x) + \mathcal{O}(\epsilon^3) \\
\psi_\epsilon(x) &= \epsilon\sin(x) + \frac{\epsilon^2}{2}\sin(2x) + \mathcal{O}(\epsilon^3)\,, \\
c_\epsilon &= 1 + \frac{1}{2}\epsilon^2 + \mathcal{O}(\epsilon^3)\,.
\end{aligned}
$$

Extension:

• **Periodic 2D traveling waves:**

- *vorticity:* Dubreil-Jacotin '34, Goyon '58, Zeidler '73, Wahlen '09, Martin '13

- *large amplitude:* Krasovskii '71, Keady-Norbury '78, Toland '78, McLeod '97, Constantin-Strauss '04 , Constantin -Strauss -Varvaruca '18

• **2D time quasi-periodic traveling waves:** Berti-Franzoi-M. '20, Berti-Franzoi-M. '21, Feola-Giuliani '20

Are Stokes waves stable/unstable under long wave perturbations?

$$\begin{cases} \eta_t = c\eta_x + G(\eta)\psi \\ \psi_t = c\psi_x - g\eta - \dfrac{\psi_x^2}{2} + \dfrac{1}{2(1+\eta_x^2)}\big(G(\eta)\psi + \eta_x\psi_x\big)^2 \end{cases}$$

Linearized water waves equations in moving frame at Stokes waves

$$h_t = \mathcal{L}_\epsilon h$$

$\mathcal{L}_\epsilon =$ linear autonomous operator with 2π-periodic coefficients

unstable "long wave" solutions

Look for solutions

$$h(t,x) = \mathrm{Re}\big(e^{\lambda t} e^{\mathrm{i}\mu x} v(x)\big)\,, \quad \mu \in \mathbb{R}\,,$$

where $v(x)$ is 2π-periodic, μ is **Floquet exponent**, and λ with strictly positive real part.

Bloch-Floquet theory

Analyze the spectrum of

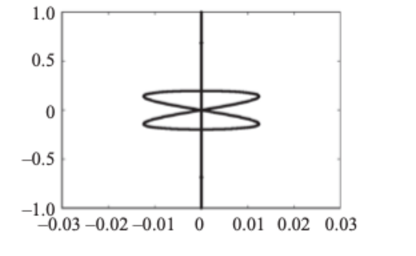
$$\mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\mu x} \circ \mathcal{L}_\epsilon \circ e^{\mathrm{i}\mu x}$$

acting on 2π-periodic functions, for $0 \le \mu \le \frac{1}{2}$.

IF $\lambda \in \sigma(\mathcal{L}_{\mu,\epsilon})$ has positive real part $\Rightarrow h(t,x)$ grows exponentially in time

- **Numerical:** Deconinck-Oliveras 2011:

Fix $\epsilon > 0$, then $\sigma(\mathcal{L}_{\mu,\epsilon})$ is *numerically* computed as μ changes: "figure 8"



- **Analytic:** Nguyen-Strauss 2020:

There exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$, there exists $\mu_0 = \mu_0(\epsilon) > 0$ such that for all $0 < |\mu| < \mu_0$, $\mathcal{L}_{\mu,\epsilon}$ has 2 eigenvalues of the form

$$\lambda^{\pm}(\mu,\epsilon) = \begin{cases} \frac{1}{\sqrt{2}}\mathrm{i}\mu \pm \frac{1}{2\sqrt{2}}\mu\epsilon + O(\mu^2) + O(\mu\epsilon^2) & \text{if } \mu > 0 \\ \frac{1}{\sqrt{2}}\mathrm{i}\mu \mp \frac{1}{2\sqrt{2}}\mu\epsilon + O(\mu^2) + O(\mu\epsilon^2) & \text{if } \mu < 0 \end{cases}$$

Main result

Theorem (Berti - M. - Ventura, 2021)

There exist $\epsilon_0, \mu_0 > 0$ such that, $\forall (\mu,\epsilon) \in [0,\mu_0) \times [0,\epsilon_0)$, the operator $\mathcal{L}_{\mu,\epsilon}$ has 4 eigenvalues close to 0 and

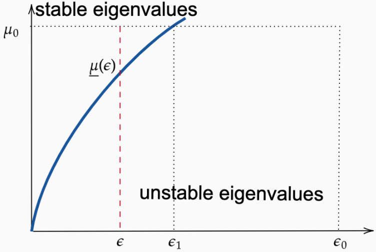
- *2 eigenvalues $\lambda_1^{\pm}(\mu,\epsilon)$ have the form*

$$\begin{cases} \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \frac{\mu}{8}\sqrt{8\epsilon^2(1+r_0(\epsilon,\mu)) - \mu^2(1+r_0'(\epsilon,\mu))}\,, & 0 \le \mu < \underline{\mu}(\epsilon) \\ \frac{1}{2}\mathrm{i}\underline{\mu}(\epsilon) + \mathrm{i}r(\epsilon^3)\,, & \mu = \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2(1+r_0'(\epsilon,\mu)) - 8\epsilon^2(1+r_0(\epsilon,\mu))}\,, & \mu > \underline{\mu}(\epsilon)\,, \end{cases}$$

where $\underline{\mu}(\epsilon) = 2\sqrt{2}\epsilon(1+r(\epsilon))$. The function $8\epsilon^2(1+r_0(\epsilon,\mu)) - \mu^2(1+r_0'(\epsilon,\mu)) > 0$, respectively < 0, for $0 < \mu < \underline{\mu}(\epsilon)$, respectively $\mu > \underline{\mu}(\epsilon)$.

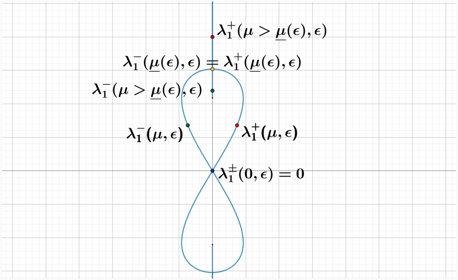
- *2 eigenvalues are purely imaginary*

Notation:
$|r(\epsilon^{m_1}\mu^{n_1}, \epsilon^{m_2}\mu^{n_2})| \le C\sum_{j=1}^{2} |\epsilon|^{m_j}|\mu|^{n_j}$
real analytic function



Curves of $\lambda^{\pm}(\mu,\epsilon) \in \mathbb{C}$ at fixed ϵ, as μ varies

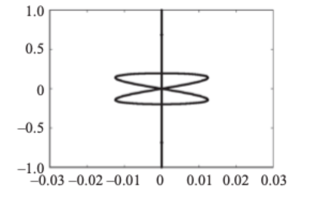
$$\lambda^{\pm}(\mu,\epsilon) \approx \begin{cases} \frac{1}{2}\mathrm{i}\mu \pm \frac{\mu}{8}\sqrt{8\epsilon^2-\mu^2}\,, & 0 \le \mu < \underline{\mu}(\epsilon) \\ \frac{1}{2}\mathrm{i}\underline{\mu}(\epsilon)\,, & \mu = \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2-8\epsilon^2}\,, & \mu > \underline{\mu}(\epsilon) \end{cases}$$



- For $0 < \mu < \underline{\mu}(\epsilon)$, $\lambda^{\pm}(\mu,\epsilon)$ have opposite non-zero real part
- As $\mu \to \underline{\mu}(\epsilon)$, the $\lambda^{\pm}(\mu,\epsilon)$ collide on $\mathrm{i}\mathbb{R}$ *far* from 0,
- For $\mu > \underline{\mu}(\epsilon)$ the $\lambda^{\pm}(\mu,\epsilon)$ are purely imaginary

Remarks

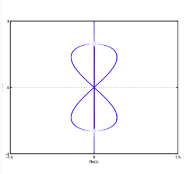
1. Our theorem describes ALL the eigenvalues close to 0, for (μ, ϵ) small
2. Complete accordance with numerical simulations by Deconinck-Oliveras '11



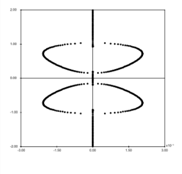
3. Nguyen-Strauss describes the unstable eigenvalues $|\mu| \ll \epsilon$, namely the cross amid the "8". We extend these local branches to global ones

4. The eigenvalues $\lambda^{\pm}(\mu, \epsilon)$ are *not analytic* in (μ, ϵ) close to $(\underline{\mu}(\epsilon), \epsilon)$. In previous approaches the eigenvalues are a-priori supposed to be analytic in (μ, ϵ). The $\lambda^{\pm}(\mu, \epsilon)$ are eigenvalues of a 2×2 matrix *analytic* in (μ, ϵ).

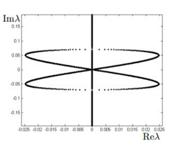
5. "Figure 8" is found *numerically* in many other models: we believe our method extends to these cases



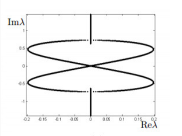
gKdV (Haragus-Kapitula)



NLS (Haragus-Kapitula)



Whitham (Deconinck Trichtchenko)



SG (Deconinck Trichtchenko)

Ideas of the proof

Difficulties:

- bifurcation problem from the defective eigenvalue 0
- the eigenvalues are not analytic “at the top of the 8”

Main ingredients:

1. Kato’s similarity transformation theory
2. exploit Hamiltonian and reversibility structure
3. "KAM inspired” block diagonalization procedure

Preparation

1. Linearize the WW equations at the Stokes wave
2. Apply two changes of coordinates:
    - linear good unknown of Alinhac
    - Levi-Civita transformation

We get the system $h_t = \mathcal{L}_\epsilon h$

$\mathcal{L}_\epsilon$ Hamiltonian: $$\mathcal{L}_\epsilon = \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=:\mathcal{J}} \underbrace{\begin{bmatrix} 1 + a_\epsilon(x) & -(1+p_\epsilon)(x)\partial_x \\ \partial_x \circ (1 + p_\epsilon(x)) & |D| \end{bmatrix}}_{=:\mathcal{B}_\epsilon}$$

$\mathcal{L}_\epsilon$ reversible: $$\mathcal{L}_\epsilon \circ \rho = -\rho \circ \mathcal{L}_\epsilon \,,$$

where
$$p_\epsilon(x) = -2\epsilon\cos(x) + \epsilon^2\Big(\frac{3}{2} - 2\cos(2x)\Big) + \mathcal{O}(\epsilon^3)$$
$$a_\epsilon(x) = -2\epsilon\cos(x) + \epsilon^2\big(2 - 2\cos(2x)\big) + \mathcal{O}(\epsilon^3)$$

The linear operator $\mathcal{L}_\epsilon$ is autonomous, pseudodifferential, Hamiltonian and reversible, and has 2π-periodic coefficients

Bloch-Floquet expansion

Use Bloch-Floquet theory to study its spectrum

$$\sigma_{L^2(\mathbb{R})}(\mathcal{L}_\epsilon) = \bigcup_{\mu \in [-\frac{1}{2}, \frac{1}{2})} \sigma_{L^2(\mathbb{T})}(\mathcal{L}_{\mu,\epsilon}), \quad \mathcal{L}_{\mu,\epsilon} := e^{-\mathrm{i}\mu x} \mathcal{L}_\epsilon \, e^{\mathrm{i}\mu x}$$

where $\mathcal{L}_{\mu,\epsilon}$ acts on $L^2(\mathbb{T}, \mathbb{C}^2)$

- $A_\mu := e^{-\mathrm{i}\mu x}\mathrm{Op}(a(x,\xi))e^{\mathrm{i}\mu x} = \mathrm{Op}(a(x,\xi+\mu))$
- $\sigma(\mathcal{L}_{-\mu,\epsilon}) = \overline{\sigma(\mathcal{L}_{\mu,\epsilon})} \implies \mu > 0$
- $\sigma(\mathcal{L}_{\mu,\epsilon})$ is 1-periodic in $\mu \implies \mu \in [-\frac{1}{2}, \frac{1}{2})$

$\Rightarrow$ We restrict to study $\sigma(\mathcal{L}_{\mu,\epsilon})$ for $\mu \in [0, \frac{1}{2})$

If λ is an eigenvalue of $\mathcal{L}_{\mu,\epsilon}$ with eigenvector $v(x)$, then

$$h(t,x) = e^{\lambda t} e^{\mathrm{i}\mu x} v(x) \quad \text{solves} \quad h_t = \mathcal{L}_\epsilon h$$

$\mathcal{L}_{\mu,\epsilon}$ is the complex *Hamiltonian* and *reversible* operator

$$\mathcal{L}_{\mu,\epsilon} = \underbrace{\begin{bmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{bmatrix}}_{=\mathcal{J}} \underbrace{\begin{bmatrix} 1 + a_\epsilon(x) & -(1+p_\epsilon(x))(\partial_x + \mathrm{i}\mu) \\ (\partial_x + \mathrm{i}\mu) \circ (1 + p_\epsilon(x)) & |D+\mu| \end{bmatrix}}_{=:\, \mathcal{B}_{\mu,\epsilon}}$$

- **(Hamiltonian)** $\mathcal{B}_{\mu,\epsilon} = \mathcal{B}^*_{\mu,\epsilon}$
- **(Reversibility preserving)** $\mathcal{B}_{\mu,\epsilon}$ commutes with

$$\bar{\rho} \begin{bmatrix} \eta(x) \\ \psi(x) \end{bmatrix} := \begin{bmatrix} \bar{\eta}(-x) \\ -\bar{\psi}(-x) \end{bmatrix}$$

Goal: describe the spectrum of $\mathcal{L}_{\mu,\epsilon}$ on $L^2(\mathbb{T})$ when (μ, ϵ) small

- start from the unperturbed spectrum of $\mathcal{L}_{0,0}$
- switch on the parameters (μ, ϵ)

The unperturbed spectrum of $\mathcal{L}_{0,0}$

$$\mathcal{L}_{0,0} = \begin{bmatrix} \partial_x & |D| \\ -1 & \partial_x \end{bmatrix}$$

- $\sigma(\mathcal{L}_{0,0})$ consists of the purely imaginary eigenvalues

$\lambda_k^{\pm}(0,0) := \mathrm{i}(k \mp \sqrt{|k|})\,, \quad k \in \mathbb{Z}\,.$

- 0 is isolated eigenvalue of $\mathcal{L}_{0,0}$ with algebraic multiplicity 4

$\lambda_0^{+}(0,0) = \lambda_0^{-}(0,0) = \lambda_1^{+}(0,0) = \lambda_{-1}^{-}(0,0) = 0$

- 0 has geometric multiplicity 3. A real basis of Kernel of $\mathcal{L}_{0,0}$ is

$$f_1^+ := \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}, \quad f_1^- := \begin{bmatrix} -\sin(x) \\ \cos(x) \end{bmatrix}, \qquad f_0^- := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

together with the generalized eigenvector

$$f_0^+ := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathcal{L}_{0,0} f_0^+ = -f_0^-$$

We want to bifurcate from the *defective* eigenvalue 0

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Kato's theory of similarity transformations

how to prolong analytically a basis of the unperturbed spectral space
to a basis of the perturbed one

Kato's theory of similarity transformations: projectors

Define the *projectors*

$$P_{\mu,\epsilon} := -\frac{1}{2\pi \mathrm{i}} \oint_\Gamma (\mathcal{L}_{\mu,\epsilon} - \lambda)^{-1} d\lambda$$

- well defined, bounded $L^2 \to H^1$, commuting with $\mathcal{L}_{\mu,\epsilon}$
- analytic in (μ, ϵ)
- $\mathcal{V}_{\mu,\epsilon} := \mathrm{Rg}(P_{\mu,\epsilon})$ is an invariant subspace

$$\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$$

  and one has the direct sum decomposition $H^1 = \mathcal{V}_{\mu,\epsilon} \oplus \mathrm{Ker}(P_{\mu,\epsilon})$
- $\sigma(\mathcal{L}_{\mu,\epsilon}) \cap \{z \in \mathbb{C} \text{ inside } \Gamma\} = \sigma(\mathcal{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}})$

Goal: Construct a basis of $\mathcal{V}_{\mu,\epsilon}$ and represent the action of $\mathcal{L}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ over this basis as a finite matrix

Q: How to do construct such a basis, in an *analytic* way?

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Kato's theory of similarity transformations: transformation operators

Define the transformation operators

$$U_{\mu,\epsilon} := \left(\mathrm{Id} - (P_{\mu,\epsilon} - P_{0,0})^2\right)^{-1/2}\left[P_{\mu,\epsilon}P_{0,0} + (\mathrm{Id} - P_{\mu,\epsilon})(\mathrm{Id} - P_{0,0})\right]$$

- well defined, bounded $H^1 \to H^1$, invertible, analytic in (μ, ϵ)
- reversibility preserving and symplectic:

$$\bar{\rho}U_{\mu,\epsilon} = U_{\mu,\epsilon}\bar{\rho}, \qquad U^*_{\mu,\epsilon}\mathcal{J}U_{\mu,\epsilon} = \mathcal{J}$$

- conjugate the spectral projectors:

$$U_{\mu,\epsilon}P_{0,0}U^{-1}_{\mu,\epsilon} = P_{\mu,\epsilon}\,, \qquad U^{-1}_{\mu,\epsilon}P_{\mu,\epsilon}U_{\mu,\epsilon} = P_{0,0}$$

- the subspaces $\mathcal{V}_{\mu,\epsilon} = \mathrm{Rg}(P_{\mu,\epsilon})$ are isomorphic one to each other:

$$\mathcal{V}_{\mu,\epsilon} = U_{\mu,\epsilon}\mathcal{V}_{0,0}$$

Transform the unperturbed basis $\{f_1^+, f_1^-, f_0^+, f_0^-\}$ of $\mathcal{V}_{0,0}$ via $U_{\mu,\epsilon}$:

Kato basis of $\mathcal{V}_{\mu,\epsilon}$, $\dim \mathcal{V}_{\mu,\epsilon} = \dim \mathcal{V}_{0,0} = 4$, for any (μ, ϵ)

$$U_{\mu,\epsilon}f_1^+ \;,\; U_{\mu,\epsilon}f_1^- \;,\; U_{\mu,\epsilon}f_0^+ \;,\; U_{\mu,\epsilon}f_0^- \,.$$

$\Rightarrow \{U_{\mu,\epsilon}f_1^{\pm},\ U_{\mu,\epsilon}f_0^{\pm}\}$ is a symplectic and reversible basis of $\mathcal{V}_{\mu,\epsilon}$

reversible $\bar{\rho}\mathtt{f}_1^+ = \mathtt{f}_1^+, \quad \bar{\rho}\mathtt{f}_1^- = -\mathtt{f}_1^-, \quad \bar{\rho}\mathtt{f}_0^+ = \mathtt{f}_0^+, \quad \bar{\rho}\mathtt{f}_0^- = -\mathtt{f}_0^-$

Next goal: Represent $\mathcal{L}_{\mu,\epsilon} \colon \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ on the basis $U_{\mu,\epsilon} f_k^\sigma$, $\sigma = \pm$, $k = 0, 1$

Lemma

The 4×4 matrix that represents the Hamiltonian and reversible operator $\mathcal{L}_{\mu,\epsilon} = \mathcal{J}\mathcal{B}_{\mu,\epsilon} : \mathcal{V}_{\mu,\epsilon} \to \mathcal{V}_{\mu,\epsilon}$ with respect to the symplectic and reversible basis $\{f_k^\sigma(\mu,\epsilon)\}_{\sigma,k}$ is

$$\mathtt{J}_4 \mathtt{B}_{\mu,\epsilon}\,, \qquad \mathtt{J}_4 := \left(\begin{array}{c|c} \mathtt{J}_2 & 0 \\ \hline 0 & \mathtt{J}_2 \end{array}\right), \quad \mathtt{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $\mathtt{B}_{\mu,\epsilon} = \mathtt{B}_{\mu,\epsilon}^*$ has matrix elements

$$\left(\mathcal{B}_{\mu,\epsilon} U_{\mu,\epsilon} f_k^\sigma\,,\, U_{\mu,\epsilon} f_{k'}^{\sigma'}\right)_{L^2}$$

The entries of the matrix $\mathtt{B}_{\mu,\epsilon}$ are alternatively real or purely imaginary

Rk 1: $U_{\mu,\epsilon}$ very explicit, can compute it perturbatively!
Rk 2: At $\mu = 0$ use the full spectral structure of $\mathcal{L}_{0,\epsilon}$, $\forall \epsilon > 0$: 0 is eigenvalue with algebraic multiplicity 4 and geometric multiplicity 2

Lemma: Matrix expansion

In a symplectically modified basis of $\mathcal{V}_{\mu,\epsilon}$ obtained from $\{U_{\mu,\epsilon} f_k^\sigma\}_{k=0,1,\sigma=\pm}$, the operator $\mathcal{L}_{\mu,\epsilon}|_{\mathcal{V}_{\mu,\epsilon}}$ is represented by the Hamiltonian and reversible matrix

$$\mathrm{L}_{\mu,\epsilon} = \mathrm{J}_4 \begin{pmatrix} E & F \\ F^* & G \end{pmatrix} \equiv \begin{pmatrix} \mathrm{J}_2 E & \mathrm{J}_2 F \\ \mathrm{J}_2 F^* & \mathrm{J}_2 G \end{pmatrix},$$

where $E = E^*, F, G = G^*$ are the 2×2 matrices

$$E := \begin{pmatrix} \epsilon^2(1 + r_1'(\epsilon, \mu\epsilon^2)) - \frac{\mu^2}{8}(1 + r_1''(\epsilon, \mu)) & -\mathrm{i}\left(\frac{1}{2}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) \\ \mathrm{i}\left(\frac{1}{2}\mu + r_2(\mu\epsilon^2, \mu^2\epsilon, \mu^3)\right) & -\frac{\mu^2}{8}(1 + r_5(\epsilon, \mu)) \end{pmatrix}$$

$$G := \begin{pmatrix} 1 + r_8(\epsilon^3, \mu^2\epsilon, \mu\epsilon^2, \mu^3) & -\mathrm{i}\mu - \mathrm{i}r_9(\mu\epsilon^2, \mu^2\epsilon, \mu^3) \\ \mathrm{i}\mu + \mathrm{i}r_9(\mu\epsilon^2, \mu^2\epsilon, \mu^3) & \mu + r_{10}(\mu^2\epsilon, \mu^3) \end{pmatrix}$$

$$F = \begin{pmatrix} r_3(\epsilon^3, \mu\epsilon^2, \mu^2\epsilon, \mu^3) & \mathrm{i}r_4(\mu\epsilon, \mu^3) \\ \mathrm{i}r_6(\mu\epsilon, \mu^3) & r_7(\mu^2\epsilon, \mu^3) \end{pmatrix}$$

Rk1: because of the Hamiltonian and reversible structure

$$E = \begin{pmatrix} \alpha & -\mathrm{i}\beta \\ \mathrm{i}\beta & \gamma \end{pmatrix}, \qquad \alpha, \beta, \gamma \in \mathbb{R} \quad \Rightarrow \quad \mathrm{J}_2 E = \mathrm{i}\beta + \begin{pmatrix} 0 & \gamma \\ -\alpha & 0 \end{pmatrix}$$

real eivenvalues IFF $-\alpha$ and γ have same sign!

Rk2: NOT a simple Taylor expansion: we need γ to have sign, so at $\mu = 0$ need that in γ there are NO terms
- $\mathcal{O}(\epsilon^k)$ for any k: exploit Jordan form of $\mathcal{L}_{0,\epsilon}$ $\forall \epsilon > 0$
- $\mathcal{O}(\mu\epsilon^k)$ for any k: slightly modify the basis

- The top-left 2×2 block of $\mathrm{L}_{\mu,\epsilon}$ shows the BF phenomenon

$$\mathrm{J}_2 E = \mathrm{i}\big(\frac{1}{2}\mu + r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\big)\mathrm{Id} + \begin{pmatrix} 0 & -\frac{\mu^2}{8}(1+r_5(\epsilon,\mu)) \\ -\epsilon^2(1+r_1'(\epsilon,\mu\epsilon^2)) + \frac{\mu^2}{8}(1+r_1''(\epsilon,\mu)) & 0 \end{pmatrix}.$$

Its eigenvalues are

$$\begin{cases} \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \frac{\mu}{8}\sqrt{8\epsilon^2(1+r_0(\epsilon,\mu)) - \mu^2(1+r_0'(\epsilon,\mu))}\,, & 0\le\mu<\tilde{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\tilde{\mu}(\epsilon) + \mathrm{i}r_2(\epsilon^3)\,, & \mu=\tilde{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2(1+r_0'(\epsilon,\mu)) - 8\epsilon^2(1+r_0(\epsilon,\mu))}\,, & \mu>\tilde{\mu}(\epsilon)\,. \end{cases}$$

- Instead $\mathrm{J}_2 G$ has purely imaginary eigenvalues of size $\mathcal{O}(\sqrt{\mu})$

- However

$$\mathrm{L}_{\mu,\epsilon} = \begin{pmatrix} \mathrm{J}_2 E & \mathrm{J}_2 F \\ \mathrm{J}_2 F^* & \mathrm{J}_2 G \end{pmatrix} \qquad \text{NOT BLOCK-DIAGONAL}$$

Idea: **Look for a perturbative block-decoupling**, cfr. KAM theory

Look for a symplectic and reversibility preserving transformation Φ s.t.

$$\Phi^{-1}\begin{pmatrix} \mathrm{J}_2 E & \mathrm{J}_2 F \\ \mathrm{J}_2 F^* & \mathrm{J}_2 G \end{pmatrix}\Phi = \begin{pmatrix} \mathrm{J}_2 E_{new} & 0 \\ 0 & \mathrm{J}_2 G_{new} \end{pmatrix}$$

and $\mathrm{J}_2 E_{new}$ with the same structure as $\mathrm{J}_2 E$

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$$\begin{cases} \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \frac{\mu}{8}\sqrt{8\epsilon^2\big(1+r_0(\epsilon,\mu)\big) - \mu^2\big(1+r_0'(\epsilon,\mu)\big)}\,, & 0\le\mu<\tilde{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\tilde{\mu}(\epsilon) + \mathrm{i}r_2(\epsilon^3)\,, & \mu=\tilde{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2\big(1+r_0'(\epsilon,\mu)\big) - 8\epsilon^2\big(1+r_0(\epsilon,\mu)\big)}\,, & \mu>\tilde{\mu}(\epsilon)\,. \end{cases}$$

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Block-decoupling

- After a first transformation which removes the element F_{11} of F which has size ϵ^3, we have

$$\mathrm{L}^{(1)}_{\mu,\epsilon} := \begin{pmatrix} \mathrm{J}_2 E^{(1)} & \mathrm{J}_2 F^{(1)} \\ \mathrm{J}_2 (F^{(1)})^* & \mathrm{J}_2 G^{(1)} \end{pmatrix}, \qquad \mathrm{J}_2 F^{(1)} = \begin{pmatrix} \mathrm{i} r_6(\mu\epsilon, \mu^3) & r_7(\mu^2\epsilon, \mu^3) \\ 0 & -\mathrm{i} r_4(\mu\epsilon, \mu^3) \end{pmatrix} = \mathcal{O}(\mu\epsilon, \mu^3)$$

It exploit the Jordan structure of $\mathrm{L}_{0,\epsilon}$

- look for an Hamiltonian and reversibility preserving matrix S such that

$$\mathrm{L}^{(2)}_{\mu,\epsilon} = e^S\, \mathrm{L}^{(1)}_{\mu,\epsilon}\, e^{-S} = \begin{pmatrix} \mathrm{J}_2 E^{(2)} & \mathrm{J}_2 F^{(2)} \\ \mathrm{J}_2 (F^{(2)})^* & \mathrm{J}_2 G^{(2)} \end{pmatrix}$$

and $E^{(2)} \sim E, \qquad \|\mathrm{J}_2 F^{(2)}\| \ll \|\mathrm{J}_2 F^{(1)}\|$

- Lie expansion: $\qquad \mathrm{L}^{(2)}_{\mu,\epsilon} = \mathrm{L}^{(1)}_{\mu,\epsilon} + [S, \mathrm{L}^{(1)}_{\mu,\epsilon}] + h.o.t.$

$$= \begin{pmatrix} \mathrm{J}_2 E^{(1)} & 0 \\ 0 & \mathrm{J}_2 G^{(1)} \end{pmatrix} + \left[S, \begin{pmatrix} \mathrm{J}_2 E^{(1)} & 0 \\ 0 & \mathrm{J}_2 G^{(1)} \end{pmatrix} \right] + \begin{pmatrix} 0 & \mathrm{J}_2 F^{(1)} \\ \mathrm{J}_2 (F^{(1)})^* & 0 \end{pmatrix} + h.o.t.,$$

- After a first transformation which removes the element F_{11} of F which has size ϵ^3, we have

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Homological equation

- Choose S to solve the **homological equation**

$$\left[S, \begin{pmatrix} J_2 E^{(1)} & 0 \\ 0 & J_2 G^{(1)} \end{pmatrix}\right] + \begin{pmatrix} 0 & J_2 F^{(1)} \\ J_2 (F^{(1)})^* & 0 \end{pmatrix} = 0$$

- Take $S = J_4 \begin{pmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{pmatrix}$, then it is equivalent to the Sylvester equation

$$J_2 E^{(1)} X - X J_2 G^{(1)} = -J_2 F^{(1)}, \qquad \text{where } X := J_2 \Sigma$$

Sylvester equation

$$AX - XB = C$$

has a solution provided e.g. $\sigma(A) \subset \{z \colon |z| < \rho\}$ and $\sigma(B) \subset \{z \colon |z| > \rho\}$

Here $\sigma(J_2 E^{(1)}) = \mathcal{O}(\mu)$, $\quad \sigma(J_2 G^{(1)}) = \mathcal{O}(\sqrt{\mu})$, so OK!

Rk: differently from KAM theory, to solve the homological equation we DO NOT diagonalize $J_2 E^{(1)}$ and $J_2 G^{(1)}$, which is not even possible when $\mu \sim 2\sqrt{2}\epsilon$ since Jordan block appears.
We compute X explicitly and prove it is analytic in (μ, ϵ)

Homological equation

- Choose S to solve the **homological equation**

$$\left[S, \begin{pmatrix} J_2 E^{(1)} & 0 \\ 0 & J_2 G^{(1)} \end{pmatrix}\right] + \begin{pmatrix} 0 & J_2 F^{(1)} \\ J_2 (F^{(1)})^* & 0 \end{pmatrix} = 0$$

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We compute X explicitly and prove it is analytic in (μ, ϵ)

Lemma

The 4×4 *Hamiltonian and reversible matrix* $\mathrm{L}^{(2)}_{\mu,\epsilon} := e^{S}\mathrm{L}^{(1)}_{\mu,\epsilon}e^{-S}$ *has structure*

$$\mathrm{L}^{(2)}_{\mu,\epsilon} = \begin{pmatrix} \mathrm{J}_2 E^{(2)} & \mathrm{J}_2 F^{(2)} \\ \mathrm{J}_2 (F^{(2)})^* & \mathrm{J}_2 G^{(2)} \end{pmatrix}$$

with $E^{(2)} \sim E$*,* $G^{(2)} \sim G$ *and*

$$\mathrm{J}_2 F^{(2)} = \begin{pmatrix} \mathrm{i} r_6(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) & r_7(\mu^3\epsilon^3, \mu^4\epsilon^2, \mu^6\epsilon, \mu^8) \\ -r_3(\mu^2\epsilon^3, \mu^3\epsilon^2, \mu^5\epsilon, \mu^7) & -\mathrm{i} r_4(\mu^2\epsilon^3, \mu^4\epsilon^2, \mu^5\epsilon, \mu^7) \end{pmatrix} = \mathcal{O}(\mu^2\epsilon^2)$$

Now the size of $\mathrm{J}_2 F^{(2)}$ is sufficiently small to completely remove the off diagonal terms via a standard implicit function theorem

Lemma

There exists a 4×4 *reversibility-preserving Hamiltonian matrix* S_2 *such that* $\mathtt{L}^{(3)}_{\mu,\epsilon} := e^{S_2}\mathtt{L}^{(2)}_{\mu,\epsilon}e^{-S_2}$ *is Hamiltonian, reversible and it has structure*

$$\mathtt{L}^{(3)}_{\mu,\epsilon} = \begin{pmatrix} \mathtt{J}_2 E^{(3)} & 0 \\ 0 & \mathtt{J}_2 G^{(3)} \end{pmatrix}$$

with

$$\mathtt{J}_2 E^{(3)} := \begin{pmatrix} \mathrm{i}\big(\tfrac{1}{2}\mu + r(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\big) & -\frac{\mu^2}{8}(1+r_5(\epsilon,\mu)) \\ \frac{\mu^2}{8}(1+r_1(\epsilon,\mu)) - \epsilon^2(1+r_1'(\epsilon,\mu\epsilon^2)) & \mathrm{i}\big(\tfrac{1}{2}\mu + r(\mu\epsilon^2,\mu^2\epsilon,\mu^3)\big) \end{pmatrix}$$

$$\mathtt{J}_2 G^{(3)} := \begin{pmatrix} \mathrm{i}\mu\big(1+r_9(\epsilon^2,\mu\epsilon,\mu^2)\big) & \mu + r_{10}(\mu^2\epsilon,\mu^3) \\ -1 - r_8(\epsilon^2,\mu^2\epsilon,\mu^3) & \mathrm{i}\mu\big(1+r_9(\epsilon^2,\mu\epsilon,\mu^2)\big) \end{pmatrix}$$

The eigenvalues of $\mathtt{J}_2 E^{(3)}$ are

$$\lambda^{\pm}(\mu,e) = \begin{cases} \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \frac{\mu}{8}\sqrt{8\epsilon^2\big(1+r_0(\epsilon,\mu)\big) - \mu^2\big(1+r_0'(\epsilon,\mu)\big)}\,, & 0 \le \mu < \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\underline{\mu}(\epsilon) + \mathrm{i}r(\epsilon^3)\,, & \mu = \underline{\mu}(\epsilon)\,, \\ \frac{1}{2}\mathrm{i}\mu + \mathrm{i}r(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \mathrm{i}\frac{\mu}{8}\sqrt{\mu^2\big(1+r_0'(\epsilon,\mu)\big) - 8\epsilon^2\big(1+r_0(\epsilon,\mu)\big)}\,, & \mu > \underline{\mu}(\epsilon)\,, \end{cases}$$

The eigenvalues of $\mathtt{J}_2 G^{(3)}$ are purely imaginary

$$\lambda_0^{\pm}(\mu,\epsilon) = \pm\mathrm{i}\sqrt{\mu}\big(1+r'(\epsilon^2,\mu\epsilon,\mu^2)\big) + \mathrm{i}\mu\big(1+r_9(\epsilon^2,\mu\epsilon,\mu^2)\big)\,.$$

Finite depth case

What about stability/instability of water waves in finite depth?

- Heuristic analysis by Benjamin and Whitham predict instability whenever

$$\kappa \mathtt{h} > 1.363...$$

  for Stokes waves of wavelength $2\pi\kappa$. For us $\kappa = 1$

- rigorous analysis by Bridges-Mielke confirm this threshold and prove splitting of eigenvalues with strictly positive real part (the cross amid the 8)

- Numerical simulations by Deconick-Oliveras describe complete figure 8 for depth larger than critical Whitham-Benjamin threshold

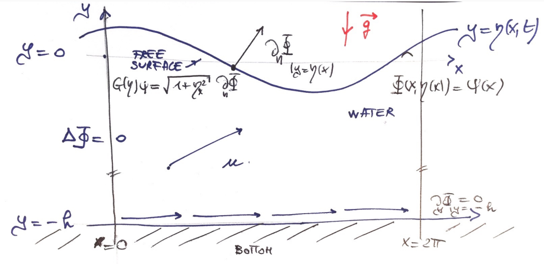
Water Waves: Euler equations for an incompressible, irrotational fluid in finite water $\mathcal{D}_\eta(t) = \{\mathtt{h} < y < \eta(t,x)\}$ under gravity.

Zakharov formulation of WW

$$\begin{cases} \eta_t = G(\eta)\psi \\ \psi_t = -g\eta - \dfrac{\psi_x^2}{2} + \dfrac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} \end{cases}$$

Unknown: $\eta(t,x)$ wave profile, $\psi(t,x) = \Phi(t,x,\eta(t,x))$ trace of velocity potential at the border

$$\begin{cases} -\Delta\Phi = 0 & \text{in } -h < y < \eta(t,x) \\ \Phi = \psi & \text{at } y = \eta(t,x) \\ \partial_y\Phi = 0 & \text{at } y = -\mathtt{h} \end{cases}$$



Dirichlet–Neumann operator: $G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n\Phi|_{y=\eta(x)} = (\Phi_y - \eta_x\Phi_x)|_{y=\eta(x)}$

- $G(\eta)$ selfadjoint, $\eta \mapsto G(\eta)$ analytic from $H^s \to H^{s-1}$
- $G(0)\psi = |D|\tanh(\mathtt{h}|D|)\psi$

We can extend our method to cover the finite depth case, and obtain

Theorem (Berti - M. - Ventura, 2022)

For any $\mathtt{h} > \mathtt{h}_{WB}$, *there exist* $\epsilon_1, \mu_0 > 0$ *such that* $\forall(\mu,\epsilon) \in (0,\mu_0) \times (0,\epsilon_1)$ *the operator* $\mathcal{L}_{\mu,\epsilon}$ *has 4 eigenvalues close to 0 and*

- *2 eigenvalues* $\lambda_1^{\pm}(\mu,\epsilon)$ *have the form*

$$\begin{cases} \mathrm{i}\frac{1}{2}\breve{\mathtt{c}}_{\mathtt{h}}\mu + \mathrm{i}r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \frac{1}{8}\mu\sqrt{\mathtt{e}_{22}(\mathtt{h})}(1+r(\epsilon,\mu))\sqrt{\Delta(\mathtt{h};\mu,\epsilon)}, & \forall\mu \in [0,\underline{\mu}(\epsilon)) \\ \mathrm{i}\frac{1}{2}\breve{\mathtt{c}}_{\mathtt{h}}\underline{\mu}(\epsilon) + \mathrm{i}r(\epsilon^3), & \mu = \underline{\mu}(\epsilon) \\ \mathrm{i}\frac{1}{2}\breve{\mathtt{c}}_{\mathtt{h}}\mu + \mathrm{i}r_2(\mu\epsilon^2,\mu^2\epsilon,\mu^3) \pm \mathrm{i}\frac{1}{8}\mu\sqrt{\mathtt{e}_{22}(\mathtt{h})}(1+r(\epsilon,\mu))\sqrt{|\Delta(\mathtt{h};\mu,\epsilon)|}, & \forall\mu \in (\underline{\mu}(\epsilon),\mu_0) \end{cases} \quad (1)$$

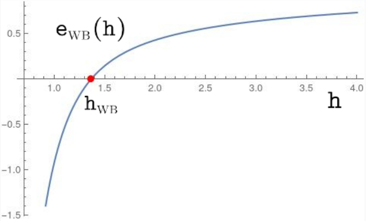
where

$$\Delta(\mathtt{h};\mu,\epsilon) := 8\mathtt{e}_{WB}(\mathtt{h})\epsilon^2 + r_1(\epsilon^3,\mu\epsilon^2) - \mathtt{e}_{22}(\mathtt{h})\mu^2(1+r_1''(\epsilon,\mu)) \,. \quad (2)$$

- *2 eigenvalues are purely imaginary*

$$\mathtt{e}_{WB}(\mathtt{h}) > 0 \Leftrightarrow \mathtt{h} > 1.363...$$

- for any $\mathtt{h} > 1.363...$ there are two eigenvalues with nonzero real part, depicting a figure 8 as μ changes
- for any $0 < \mathtt{h} < 1.363...$ all the eigenvalues are purely imaginary



Thanks for your attention!

alberto.maspero@sissa.it