Blow-up analysis of a curvature prescription problem in the disk

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The Kazdan-Warner problem '74

Let (Σ, g) be a compact surface Σ with metric g.

Question: Is a given function K the Gaussian curvature of a new metric $\tilde{g}$ conformal to g?

$\rightsquigarrow \tilde{g} = e^u g$ — e^u: conformal factor

K_g : Gaussian curvature of the metric g

$$\Longleftrightarrow \quad -\Delta u + 2K_g(x) = 2K(x)e^u \text{ in } \Sigma$$

By the Gauss-Bonnet theorem

$$\int_\Sigma K \, dA = 2\pi \chi(\Sigma)$$

dA = element of area wrt $\tilde{g}$

$\chi(\Sigma)$ = Euler characteristic of Σ

$\Longrightarrow$

Necessary conditions:

- $\chi(\Sigma) > 0$: K positive somewhere
- $\chi(\Sigma) = 0$: K changes sign (or $K \equiv 0$)
- $\chi(\Sigma) < 0$: K negative somewhere

We focus on the case $\Sigma = S^2$ ⇝ Nirenberg's problem

$$-\Delta u + 2 = 2K(x)e^u \quad \text{in } S^2$$

⇝ K somewhere positive

* If $K(x) = K(-x)$ ⇝ ∃ sol (Moser '73)
* If u is a sol., then

$$\int_{S^2} e^u \nabla K \cdot \nabla F \, dA = 0$$

for all 1^{st} order spherical harmonics F.

(Kazdan-Warner '74)

Question: Compactness: given a sequence $\{u_n\}$ of solutions, can we pass to the limit?

⇝ L^∞ uniform boundedness (Brezis-Merle '91, Li-Shafrir '94)

Blow-up analysis

Consider $\boxed{-\Delta u_n + 2 = K_n e^{u_n} \text{ in } S^2}$ with $K_n \to K$ in $C^2(S^2)$

* If $K_n = 1$: invariant by the group of conformal maps of the sphere ⇝ non compactness
* If $K_n \neq$ constant: this invariance is lost but concentration can still occur ⇝ BUBBLING PHENOMENA (mass concentration at some point)

Theorem (Chang-Yang '91, Chang-Gursky-Yang '93)

Let $K>0$. If $\{u_n\}$ blow-up at p, i.e., if $\exists \{z_n\}$ st $u_n(z_n) \to +\infty$, $z_n \to p$, then

$$\boxed{\nabla K(p) = \Delta K(p) = 0}$$

Obs: If K st $\nabla K, \Delta K \neq 0$ ⇝ Existence

Let Σ be a surface with boundary

⇝ We would like to prescribe the geodesic curvature at $\partial\Sigma$:

$$\begin{cases} -\Delta u + 2K_g = 2Ke^u & \text{in } \Sigma \\ \frac{\partial u}{\partial \nu} + 2h_g = 2he^{u/2} & \text{on } \partial\Sigma \end{cases}$$

* Brendle '02: K, h constants ⇝ parabolic flow.
* López Soriano - Malchiodi - Ruiz '20: $\Sigma \neq \mathbb{D}$ and $K<0$ ⇝ variational techniques

Case $\Sigma = \mathbb{D}$ Generalization of Nirenberg's problem

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D} \\ \frac{\partial u}{\partial \nu} + 2 = 2he^{u/2} & \text{on } \partial\mathbb{D} \end{cases}$$

⇝ Noncompact group of conformal maps of the disk

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D} \\ \frac{\partial u}{\partial \nu} + 2 = 2he^{u/2} & \text{on } \partial\mathbb{D} \end{cases}$$

* $h = 0$: Chang - Yang '88
* $K = 0$: Chang - Liu '96, Liu - Huang '05, Guo - Liu '06.
* h, K non constants : Cruz - Ruiz '18 (under symmetry conditions) Cherrier '84, Hamza '90

Necessary conditions?

By the Gauss-Bonnet theorem:

$$\int_{\mathbb{D}} Ke^u + \int_{\partial\mathbb{D}} he^{u/2} = 2\pi \implies K \text{ or } h \text{ must be positive somewhere.}$$

Kazdan - Warner type condition:

Proposition

If u is a solution then

$$\int_{\mathbb{D}} e^u \nabla K \cdot F = 4 \int_{\partial\mathbb{D}} h_\tau e^{u/2} y \quad \text{where } F(x,y) = (1 - x^2 + y^2, -2xy)$$

Let u_n be a sequence of sols. to

$$\begin{cases} -\Delta u_n = 2K_n e^{u_n} & \text{in } \mathbb{D} \\ \frac{\partial u_n}{\partial \nu} + 2 = 2h_n e^{u_n/2} & \text{on } \partial\mathbb{D} \end{cases} \quad \text{with} \quad K_n \xrightarrow[C^2(\overline{\mathbb{D}})]{} K, \quad h_n \xrightarrow[C^2(\partial\mathbb{D})]{} h$$

such that

$$\underbrace{\sup\{u_n\} \longrightarrow +\infty}_{\text{blowing-up seq.}} \quad \text{and} \quad \underbrace{\int_{\mathbb{D}} e^{u_n} + \int_{\partial\mathbb{D}} e^{u_n/2} \leq C}_{\text{bded global mass condition}}$$

Theorem (Jevnikar – López Soriano – M. – Ruiz)

$\exists\,!\, p \in \partial\mathbb{D}$ such that:

(i) If $K(p) \leq 0$ then $h(p) > \sqrt{-K(p)}$

(ii) $\exists\, a_n \in \mathbb{D}$, $a_n \to p$ st $\boxed{u_n(z) = u_{a_n}(z) + \psi_n(z)}$,

with

$$u_{a_n}(z) = 2\log\left\{ \frac{2\hat{\Phi}_n(1-|a_n|^2)}{\hat{\Phi}_n^2|1-\bar{a}_n z|^2 + \hat{K}_n|z-a_n|^2} \right\},$$

$$\hat{\phi}_n := \phi_n\left(\frac{a_n}{|a_n|}\right), \quad \Phi_n(z) := h_n(z) + \sqrt{h_n(z)^2 + K_n(z)}, \quad \hat{K}_n := K_n\left(\frac{a_n}{|a_n|}\right),$$

and $\quad \|\psi_n\|_{C^{0,\alpha}(\overline{\mathbb{D}})} \longrightarrow 0 \quad \forall\, \alpha \in (0, \tfrac{1}{2})$

(iii) $\boxed{\nabla\Phi(p) = 0}$, where

$$\Phi(z) := H(z) + \sqrt{H(z)^2 + K(z)}, \quad \text{and} \quad \begin{cases} \Delta H = 0 & \text{in } \mathbb{D} \\ H = h & \text{on } \partial\mathbb{D} \end{cases}$$

(harmonic extension of h)

Obs: $\quad \nabla\Phi(p) = 0 \iff \begin{cases} H_\tau(p) = -\dfrac{K_\tau(p)}{2\Phi(p)} \\[2ex] H_\nu(p) = -\dfrac{K_\nu(p)}{2\Phi(p)} \end{cases}$

But... $\quad H_\nu(p) = (-\Delta)^{1/2} h(p)$!!

$\Rightarrow$ If u_n blows-up at p: $\quad \boxed{h_\tau(p) = -\dfrac{K_\tau(p)}{2\Phi(p)}, \quad (-\Delta)^{1/2} h(p) = -\dfrac{K_\nu(p)}{2\Phi(p)}}$

* If $h \equiv 0 \rightsquigarrow K(p) > 0, \ \nabla K(p) = 0.$
* If $K \equiv 0 \rightsquigarrow h(p) > 0, \ h_\tau(p) = (-\Delta)^{1/2} h(p) = 0$

Let us make some comments on the proof of the theorem:

(1) The singular set $S := \{ p \in \overline{\mathbb{D}} : \exists\, z_n \in \overline{\mathbb{D}}, z_n \to p \text{ st } u_n(z_n) \to +\infty \}$

Minimal mass lemma (Battaglia-López Soriano '20) + Bounded global mass condition } $\Rightarrow$ S finite

Quantization of mass:

$$K_n e^{u_n} \rightharpoonup \sum_{p \in S \cap \mathbb{D}} 4\pi \delta_p + \sum_{p \in S \cap \partial\mathbb{D}} \gamma_p \delta_p, \qquad h_n e^{u_n/2} \rightharpoonup \sum_{p \in S \cap \partial\mathbb{D}} \gamma_p' \delta_p$$

with $\gamma_p + \gamma_p' = 2\pi$, $\gamma_p = 2\pi \dfrac{h(p)}{\sqrt{h^2(p)+K(p)}}$

* Interior blow-up points ⇝ Limit problem in $\mathbb{R}^n$ (Chen-Li, '91)
* Boundary blow-up points ⇝ Limit problem in $\mathbb{R}^n_+$ (Zhang '03, Gálvez-Mira '09, Tarantello '04)

Using the Gauss-Bonnet theorem ...

$$2\pi = \int_{\mathbb{D}} K_n e^{u_n} + \int_{\partial \mathbb{D}} h_n e^{u_n/2} \longrightarrow 4\pi N + 2\pi M,$$

where $N = |S \cap \mathbb{D}|$, $M = |S \cap \partial\mathbb{D}|$.

$\Longrightarrow$ $N = 0$, $M = 1$ $\Longrightarrow$ $S = \{p\} \in \partial \mathbb{D}$

② Profile of the solution : $u_n(z) \approx 2\log\left\{ \frac{2\hat{\Phi}_n(1-|a_n|^2)}{\hat{\Phi}_n^2\,|1-\bar{a}_n z|^2} \right\}$

The function

$$v_n(z) := u_n(f_{a_n}(z)) + 2\log \frac{1-|a_n|^2}{|1+\bar{a}_n z|^2}, \qquad f_a(z) := \frac{a+z}{1+\bar{a}z}$$

conformal function

solves the problem:

$$\begin{cases} -\Delta v_n = 2K_n(f_{a_n}(z))\,e^{v_n} & \text{in } \mathbb{D} \\ \dfrac{\partial v_n}{\partial \nu} + 2 = 2h_n(f_{a_n}(z))\,e^{v_n/2} & \text{on } \partial\mathbb{D} \end{cases}$$

with $a_n \to p \in \partial\mathbb{D}$ and $\int_{\mathbb{D}} x\,e^{v_n(z)}\,dz = \int_{\mathbb{D}} y\,e^{v_n(z)}\,dz = 0$

Idea: $v_n \to v_0$ in $C^{1,\alpha}(\overline{\mathbb{D}})$ with v_0 sol. of

$$\begin{cases} -\Delta v_0 = 2K(p)\,e^{v_0} & \text{in } \mathbb{D} \\ \dfrac{\partial v_0}{\partial \nu} + 2 = 2h(p)\,e^{v_0/2} & \text{on } \partial\mathbb{D} \end{cases}, \qquad v_0(z) = 2\log\left\{ \frac{2\Phi_0}{\Phi_0^2 + K_0|z|^2} \right\}$$

③ Nonlocal condition on h

Sup. $p=1$ and $c_n = \lambda_n \in \mathbb{R}$, $\lambda_n \longrightarrow 1$.

$\rightsquigarrow \quad u_n = u_{\lambda_n} + \varphi_n$

By the Kazdan-Warner condition,

$$\left.\begin{aligned} &2\int_{\mathbb{D}} (K_n)_y e^{u_{\lambda_n}} xy + \int_{\mathbb{D}} (K_n)_x e^{u_{\lambda_n}} (1-x^2+y^2) \\ &= \boxed{-4\int_{\partial\mathbb{D}} (h_n)_\tau e^{u_{\lambda_n}/2} y} + o_n(1)(1-\lambda_n) \end{aligned}\right\} \text{(KW)}$$

(*)

$$(*) = -4\int_{\partial\mathbb{D}} \left(h_n(z) - h_n(1)\right)_\tau e^{u_{\lambda_n}/2} y$$

$$= c_n(1-\lambda_n)\int_{\partial\mathbb{D}} \left(h_n(z)-h_n(1)\right) \frac{x(1-\lambda_n)^2 + 2\lambda_n(x-1)}{\left((1-\lambda_n)^2 - 2\lambda_n(x-1)\right)^2} \quad =: f_n$$

$$f_n := \frac{x(1-\lambda_n)^2 + 2\lambda_n(x-1)}{((1-\lambda_n)^2 - 2\lambda_n(x-1))^2}$$

Let $\varepsilon > 0$ st $1-\lambda_n < \varepsilon$.

$$\int_{\partial\mathbb{D}} (h_n(z) - h_n(1)) f_n = \underbrace{\int_{\partial\mathbb{D}\cap B_{1-\lambda_n}(1)}}_{A_n} + \underbrace{\int_{\partial\mathbb{D}\cap(B_\varepsilon(1)\setminus B_{1-\lambda_n}(1))}}_{B_n} + \underbrace{\int_{\partial\mathbb{D}\setminus B_\varepsilon(1)}}_{C_n}$$

* By symmetry and regularity of h

$\exists$ $c > 0$ st $|A_n| \le c(1-\lambda_n)$, $|B_n| \le c\varepsilon$

* In $\partial\mathbb{D}\setminus B_\varepsilon(1)$ f_n is not singular

$\rightsquigarrow$ $|(h_n(z) - h_n(1)) f_n| \le c \|h\|_{L^\infty(\partial\mathbb{D})} \in L^1(\partial\mathbb{D}\setminus B_\varepsilon(1))$

$$\Rightarrow \lim_{n\to\infty} C_n = \int_{\partial\mathbb{D}\setminus B_\varepsilon(1)} \frac{h(1) - h(z)}{2(1-x)} dz = \int_{\partial\mathbb{D}\setminus B_\varepsilon(1)} \frac{h(1) - h(z)}{|1-z|^2} dz$$

Making $n \to \infty$ in (KW)...

$$\int_{\partial\mathbb{D}\setminus B_\varepsilon(1)} \frac{h(1)-h(z)}{|1-z|^2}\,dz + O(\varepsilon) = -\frac{\pi}{2\Phi(1)} K_x(1)$$

... and letting now $\varepsilon \to 0$,

$$\underbrace{\lim_{\varepsilon\to 0} \int_{\partial\mathbb{D}\setminus B_\varepsilon(1)} \frac{h(1)-h(z)}{|1-z|^2}\,dz}_{\pi(-\Delta)^{1/2}h(1)} = -\frac{\pi}{2\Phi(1)} K_x(1)$$

$$\Longrightarrow \boxed{(-\Delta)^{1/2}h(1) = -\frac{K_x(1)}{2\Phi(1)}}$$

Very recently...

Theorem (D. Ruiz, '21)

Let $K \in C^2(\overline{\mathbb{D}})$, $h \in C^2(\partial\mathbb{D})$, with

$$K \geq 0,\ h > 0 \quad \text{or} \quad K > 0,\ h \geq 0.$$

If $\nabla\Phi(x) \neq 0 \quad \forall x \in \partial\mathbb{D}$ and $\deg_B(\nabla\Phi, \mathbb{D}, 0) \neq 0$

then there exists a solution of

$$\begin{cases} -\Delta u = 2Ke^u & \text{in } \mathbb{D} \\ \dfrac{\partial u}{\partial \nu} + 2 = 2he^{u/2} & \text{on } \partial\mathbb{D} \end{cases}$$

Question: Given K, h and p satisfying $\nabla \Phi(p) = 0$, can we construct a blowing-up sequence?

Answer: Yes.

⇝ Suppose $p = 1$ and
$$K_\varepsilon(z) := K(z) + \varepsilon G(z)$$
$$h_\varepsilon(z) := h(z) + \varepsilon I(z)$$
with $G \in C^2(\overline{\mathbb{D}})$, $I \in C^2(\partial \mathbb{D})$.

Theorem (Battaglia - M. - Pistoia), part 1:

Sup. $\nabla \Phi(1) = 0$ (and other conditions). Then $\exists\, \varepsilon_0 > 0$ st $\forall\, \varepsilon \in (0, \varepsilon_0)$ or $\forall\, \varepsilon \in (-\varepsilon_0, 0)$ there exists a sol. u_ε of the problem
$$\begin{cases} -\Delta u = 2K_\varepsilon(z) e^u & \text{in } \mathbb{D} \\ \dfrac{\partial u}{\partial \nu} + 2 = 2h_\varepsilon(z) e^{u/2} & \text{on } \partial \mathbb{D}, \end{cases}$$
blowing-up at 1 as $\varepsilon \to 0$.

Technique: Lyapunov-Schmidt reduction.

Let $f = f_{\delta,\xi}(z) = \dfrac{z + (1-\delta)\xi}{1 + (1-\delta)\bar{\xi} z}$, $\quad \xi \in \partial\mathbb{D}$, $\xi = \xi_\varepsilon \longrightarrow 1$

$(\xi = e^{i\eta}$, $\eta = \eta_\varepsilon \longrightarrow 0)$

$\delta = \delta_\varepsilon \longrightarrow 0$, $\quad \delta > 0$

$\rightsquigarrow$ $(1-\delta)\xi \longrightarrow 1$, $\quad (1-\delta)\xi \in \mathbb{D}$

If u is a sol. of (P_ε), then

$$v(z) = u(f(z)) + 2\log|f'(z)|$$

solves

$$\begin{cases} -\Delta v = 2K_\varepsilon(f(z))e^{v} & \text{in } \mathbb{D} \\ \dfrac{\partial v}{\partial \nu} + 2 = 2h_\varepsilon(f(z))e^{v/2} & \text{on } \partial\mathbb{D} \end{cases} \qquad (P_\varepsilon^*)$$

Goal: Find an approximation V_* such that

$$v = V_* + \varphi$$

solves the problem for some small function φ.

How do we choose V_*?

First attempt: $V_* = V_\zeta(z) = 2\log\left(\dfrac{\phi(\zeta)}{\phi(\zeta)^2 + K(\zeta)|z|^2}\right)$,

that is a sol. of

$$\begin{cases} -\Delta v = 2K(\zeta)\, e^{v} & \text{in } \mathbb{D} \\ \dfrac{\partial v}{\partial \nu} + 2 = 2h(\zeta)\, e^{v/2} & \text{on } \partial\mathbb{D} \end{cases}$$

⇝ Too large error ⇝ we need to refine the approximation

Second attempt: $\boxed{V_* = V_\zeta(z) + W_\zeta(z) + \tau}$, where

$$W_\zeta(z) = -\frac{2}{\pi}\int_{\partial\mathbb{D}} \log|z-w|\,\big(h(f(w)) - h(\zeta)\big)\,dw$$

solves

$$\begin{cases} -\Delta W = 0 & \text{in } \mathbb{D} \\ \dfrac{\partial W}{\partial \nu} = 2\big(h(f(z)) - h(\zeta)\big)e^{V/2} - \dfrac{1}{\pi}\displaystyle\int_{\partial\mathbb{D}} \big(h(f(z)) - h(\zeta)\big)e^{V/2} & \text{on } \partial\mathbb{D} \end{cases}$$

and $\tau = \tau_\varepsilon \longrightarrow 0$.

Obs: $W(z) = O\left(\delta\left(1+\log\frac{1}{|z+\xi|}\right)\right)$ ⇝ order δ in L^p

From here, usual **Lyapunov-Schmidt scheme**:

$v = V_* + \varphi$ sol of (P_ε^*) <⟹> φ is a sol. of

$$\begin{cases} \mathcal{L}_0^{int}\varphi := -\Delta\varphi - 2K(\xi)e^V\varphi = \mathcal{E}^{int} + \mathcal{L}^{int}\varphi + \mathcal{N}^{int}(\varphi) & \text{in } \mathbb{D} \\ \mathcal{L}_0^{\partial}\varphi := \frac{\partial\varphi}{\partial\nu} - h(\xi)e^{V/2}\varphi = \mathcal{E}^{\partial} + \mathcal{L}^{\partial}\varphi + \mathcal{N}^{\partial}(\varphi) & \text{on } \partial\mathbb{D} \end{cases}$$

(linearized operator; error term; small linear term; nonlinear term)

Linear theory + Fixed point argument ⇝ we solve

$$\begin{cases} \mathcal{L}_0^{int}\varphi = \mathcal{E}^{int} + \mathcal{L}^{int}\varphi + \mathcal{N}^{int}(\varphi) + c_0 + 2K(\xi)e^V(c_1\mathcal{Z}_1 + c_2\mathcal{Z}_2) \\ \mathcal{L}_0^{\partial}\varphi = \mathcal{E}^{\partial} + \mathcal{L}^{\partial}\varphi + \mathcal{N}^{\partial}(\varphi) + c_0 + h(\xi)e^{V/2}(c_1\mathcal{Z}_1 + c_2\mathcal{Z}_2) \end{cases}$$

where $c_0, c_1, c_2 \in \mathbb{R}$ and $\ker(\mathcal{L}_0) = \text{span}\{\mathcal{Z}_1, \mathcal{Z}_2\}$.

Adjusting $\delta = O\left(\frac{|\varepsilon|}{\log\frac{1}{|\varepsilon|}}\right)$, $\tau = O\left(\frac{|\varepsilon|}{\log\frac{1}{|\varepsilon|}}\right)$, $\eta = O(|\varepsilon|)$

we may have $c_0 = c_1 = c_2 = 0$ $\Rightarrow$ $v = V_* + \varphi$ sol of (P_ε^*)

Theorem, part 2:

$\exists\ \delta_\varepsilon > 0$ and $\zeta_\varepsilon \in \partial\mathbb{D}$ with $\delta_\varepsilon = O\left(\frac{|\varepsilon|}{\log\frac{1}{|\varepsilon|}}\right)$ and $\zeta_\varepsilon = 1 + O(|\varepsilon|)$ such that

$$u_\varepsilon\left(f_{\delta_\varepsilon,\zeta_\varepsilon}(z)\right) + 2\log|f'_{\delta_\varepsilon,\zeta_\varepsilon}(z)| - 2\log\left(\frac{2\Phi(1)}{\Phi(1)^2 + K(1)|z|^2}\right) = O\left(\frac{|\varepsilon|}{\log\frac{1}{|\varepsilon|}}\right)$$

in $L^p(\mathbb{D})$.

Obs: We need some nondegeneracy assumptions:

$$\Delta K(1) + 4|\nabla H(1)|^2 \neq 0$$

$$\partial_{22} K(1) + 2\,\Phi(1)\,h_{\tau\tau}(1) \neq 0$$

If $K \equiv 0$ $\rightsquigarrow$ contradiction with $\nabla\Phi(1) = 0$!!!

Case $K \equiv 0$:

Nirenberg problem:

$$-\Delta u + 2 = 2K(x)e^u \quad \text{in } \mathbb{S}^2$$

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{D} \\ \dfrac{\partial u}{\partial \nu} + 2 = 2h e^{u/2} & \text{on } \partial\mathbb{D} \end{cases} \iff (-\Delta)^{1/2} u + 2 = 2h e^{u/2} \quad \text{in } \mathbb{S}^1$$

$\leadsto$ Da Lio - Martinazzi - Rivière '15

Theorem (Battaglia - M. - Pistoia)

Sup. $h(1) > 0$, $h_\tau(1) = (-\Delta)^{1/2} h(1) = 0$ (and other conds.).
Then $\exists\, \varepsilon_0 > 0$ st $\forall \varepsilon \in (0, \varepsilon_0)$ or $\forall \varepsilon \in (-\varepsilon_0, 0)$ there exists a sol. u_ε of

$$(-\Delta)^{1/2} u + 2 = 2h_\varepsilon(z) e^{u/2} \quad \text{in } \mathbb{S}^1$$

blowing-up at 1. Furthermore, $\exists\, \delta_\varepsilon > 0$ and $\xi_\varepsilon \in \mathbb{S}^1$ with $\delta_\varepsilon = O(|\varepsilon|)$ and $\xi_\varepsilon = 1 + O(|\varepsilon|)$ such that

$$u_\varepsilon(f_{\delta_\varepsilon, \xi_\varepsilon}(z)) + \log|f'_{\delta_\varepsilon, \xi_\varepsilon}(z)| + 2\log h(1) = O(|\varepsilon|) \quad \text{in } L^p(\mathbb{S}^1).$$

¡Muchas gracias!

Merci beaucoup!

Thank you!