On the stability of the Ginzburg-Landau vortex

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The Gross-Pitaevskii equation

$$(\mathrm{GP}) : \begin{cases} i\partial_t u + \Delta u = (|u|^2 - 1)u \\ |u| \to 1 \text{ when } |x| \to \infty. \end{cases}$$

$u : \mathbb{R}^2_x \times \mathbb{R}_t \to \mathbb{C}$.

Energy of a solution:

$$E_{GP}(u) := \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(|u|^2 - 1)^2.$$

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Cauchy problem: Global well possedness in

- $1 + H^1$ (Bethuel-Saut)
- The energy space (Gerard)
- More complex spaces (Gallo, Bethuel-Smets)

The stationnary problem

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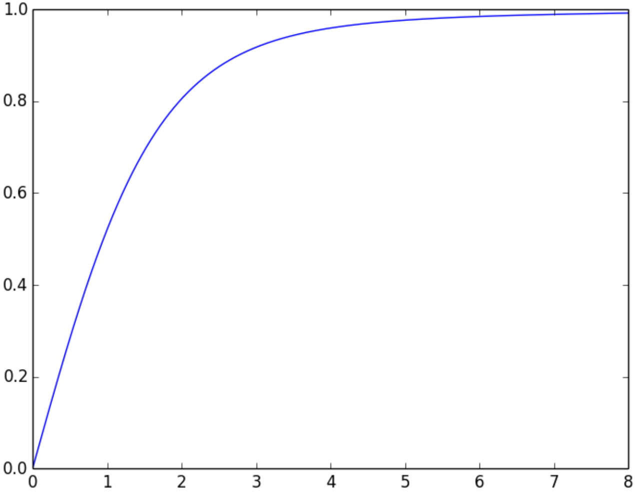
Theorem (Hervé,Hervé)

For all $n \in \mathbb{Z}^*$, there exists $V_n(x) = \rho_n(r) e^{in\theta}$ such that

$$\Delta V_n = (|V_n|^2 - 1) V_n$$

with $\rho_n(0) = 0, \rho_n(+\infty) = 1$.

Graph of ρ_1



About vortices of degrees ± 1

For $X \in \mathbb{R}^2, \gamma \in \mathbb{R}$,

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$$\text{Question : Orbital stability of } V_1.$$

Coercivity around the vortex

$(GP)(V_1+\varepsilon) = L(\varepsilon) + NL(\varepsilon)$ with
Linearized operator:

$$L_{V_1}(\varepsilon) := -\Delta\varepsilon - (1-|V_1|^2)\varepsilon + 2\mathfrak{Re}(\overline{V_1}\varepsilon)V_1,$$

Quadratic form:

$$B(\varepsilon) := \int_{\mathbb{R}^2} |\nabla\varepsilon|^2 - (1-|V_1|^2)|\varepsilon|^2 + 2\mathfrak{Re}^2(\overline{V_1}\varepsilon)$$

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Theorem (Del Pino, Felmer, Kowalczyk)

For $\varepsilon \in H$ with $||\varepsilon||_H^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1}\varepsilon)|^2 + \frac{|\nabla\varepsilon|^2}{1+r^2}$, if

$$\int_{B(0,1)} \mathfrak{Re}(\bar{\varepsilon}\partial_{x_1} V_1) = \int_{B(0,1)} \mathfrak{Re}(\bar{\varepsilon}\partial_{x_2} V_1) = \int_{B(0,1)} \mathfrak{Re}(\bar{\varepsilon} i V_1) = 0,$$

$$B(\varepsilon) \geqslant \kappa(||\varepsilon||_H^2 + \int_{\mathbb{R}^2} \mathfrak{Re}^2(\overline{V_1}\varepsilon)).$$

The norm H

$$||\varepsilon||_H^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1}\varepsilon)|^2 + \frac{|\nabla\varepsilon|^2}{1+r^2}$$

controls

$$||\chi\varepsilon||_{L^2} \leqslant K||\chi\overline{\nabla V_1}\varepsilon||_{L^2} \leqslant K(||\chi\nabla(\overline{V_1}\varepsilon)||_{L^2} + ||\chi\nabla\varepsilon||_{L^2})$$

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and by Hardy inequality,

$$\int_{\mathbb{R}^2} \frac{|V_1|^2|\varepsilon|^2}{(1+r^2)ln(2+r)^2}) \leqslant K(||\chi\varepsilon||_{L^2} + ||\nabla(\overline{V_1}\varepsilon)||_{L^2}).$$

Sketch of the proof of the coercivity

- Decompose $\varepsilon = e^{i\theta} \sum_{j\in\mathbb{Z}} \phi_j(r,\theta)$ in well choosen harmonics

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- $B_{\pm 1}(\phi_{\pm 1}) = \int_{\mathbb{R}^2} |\partial_r \phi_{\pm 1} - A(r)\phi_{\pm 1}|^2$ which cancels only if $\varepsilon = \partial_{x_{1,2}} V_1$

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- B_j for $|j| \geqslant 2$ is always coercive

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3) $B(\varepsilon) \geqslant \kappa||\varepsilon||^2$ under three orthogonality conditions
4) $NL(\varepsilon) \geqslant -K||\varepsilon||^3$
5) $1 \gg E_{GP}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) - E_{GP}(V_1) \geqslant \kappa||\varepsilon||^2(1 - K||\varepsilon||)$

The energy

$$E_{GP}(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(|u|^2 - 1)^2 = \int_{\mathbb{R}^2} e_{GP}(u).$$

We have $E_{GP}(V_1) = +\infty$ because $\nabla V_1 \sim \frac{e_\theta}{r} \notin L^2(\mathbb{R}^2)$.

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We define the renormalized energy by

$$\mathcal{E}(u) := \lim_{r \to \infty} \int_{B_r} e_{GP}(u) - e_{GP}(V_1)$$

and the space

$$E := \{\psi \in H^1_{loc}(\mathbb{R}^2), ||\psi||_H < +\infty, 1 - |\psi|^2 \in L^2(\mathbb{R}^2)\}$$

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Proposition

$\mathcal{E}$ is well defined on E and is invariant by translation and shift of phase. Furthermore, it is conserved by the flow.

Minimizing the energy

We have

$$E_{GP}(u) = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{1}{4}(|u|^2 - 1)^2 = \int_{\mathbb{R}^2} e_{GP}(u),$$

$$\mathcal{E}(u) := \lim_{r\to\infty} \int_{B_r} e_{GP}(u) - e_{GP}(V_1)$$

and

$$E := \{\psi \in H^1_{loc}(\mathbb{R}^2), ||\psi||_H < +\infty, 1 - |\psi|^2 \in L^2(\mathbb{R}^2)\}.$$

Theorem (Mironescu)

The functional $\mathcal{E}$ satisfies $\mathcal{E}(u) \geqslant 0$ for $u \in E$. Furthermore, if $\mathcal{E}(u) = 0$, then there exists $a \in \mathbb{R}^2, \gamma \in \mathbb{R}$ such that

$$u = V_1(. - a)e^{i\gamma}.$$

So far so good

1) The renormalized energy $\mathcal{E}$ is conserved by the flow
2) $\mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) = B(\varepsilon) + NL(\varepsilon)$
3) $B(\varepsilon) \geqslant \kappa||\varepsilon||^2$ under three orthogonality conditions
4) $NL(\varepsilon) \geqslant -K||\varepsilon||^3$
5) $1 \gg \mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) \geqslant \kappa||\varepsilon||^2(1 - K||\varepsilon||)$

Decomposition of the renormalized energy

We have $\mathcal{E}(V_1 + \varepsilon) = B(\varepsilon) + NL(\varepsilon)$ with

$$B(\varepsilon) = \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \varepsilon|^2 - (1 - |V_1|^2)|\varepsilon|^2 + 2\mathfrak{Re}^2(\overline{V_1}\varepsilon)$$

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and

$$NL(\varepsilon) := \int_{\mathbb{R}^2} \mathfrak{Re}(\overline{V_1}\varepsilon)|\varepsilon|^2 + \frac{1}{4}|\varepsilon|^4.$$

On a roll !

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Nonlinear terms

$$||\varepsilon||^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1}\varepsilon)|^2 + \frac{|\nabla\varepsilon|^2}{1+r^2} + \mathfrak{Re}^2(\overline{V_1}\varepsilon) + \frac{|\varepsilon|^2}{(1+r^2)ln(2+r)^2}$$

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$$\int_{\mathbb{R}^2} \mathfrak{Re}(\overline{V_1}\varepsilon)|\varepsilon|^2 \geqslant -K||\varepsilon|| \times ||\varepsilon||^2_{L^4(\mathbb{R}^2)}$$

But $||\varepsilon||_{L^4(\mathbb{R}^2)}$ can not be controlled by $||\varepsilon||$

Gloups

1) The renormalized energy $\mathcal{E}$ is conserved by the flow
2) $\mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) = B(\varepsilon) + NL(\varepsilon)$
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Remark on the nonlinear part

We have

$$NL(\varepsilon) = \int_{\mathbb{R}^2} \mathfrak{Re}(\overline{V_1}\varepsilon)|\varepsilon|^2 + \frac{1}{4}|\varepsilon|^4 = \frac{1}{4}\int_{\mathbb{R}^2} (2\mathfrak{Re}(\overline{V_1}\varepsilon) + |\varepsilon|^2)^2 - \int_{\mathbb{R}^2} \mathfrak{Re}^2(\overline{V_1}\varepsilon).$$

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We try the decomposition
$\mathcal{E}(V_1 + \varepsilon) = \tilde{B}(\varepsilon) + \tilde{NL}(\varepsilon)$ with

$$\tilde{NL}(\varepsilon) := \frac{1}{4}\int_{\mathbb{R}^2} (2\mathfrak{Re}(\overline{V_1}\varepsilon) + |\varepsilon|^2)^2.$$

A new hope

1) The renormalized energy $\mathcal{E}$ is conserved by the flow
2) $\mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) = \tilde{B}(\varepsilon) + \tilde{NL}(\varepsilon)$
3) $\tilde{B}(\varepsilon) \geqslant \kappa ||\varepsilon||^2$ under three orthogonality conditions
4) $\tilde{NL}(\varepsilon) \geqslant 0$
5) $1 \gg \mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) \geqslant \kappa ||\varepsilon||^2$

The new quadratic form

The quadratic form

$$\tilde{B}(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \varepsilon|^2 - (1 - |V_1|^2)|\varepsilon|^2$$

has infinitely many negative directions.

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Remark: $||\varepsilon||_H$ does not control $||\varepsilon||_{L^4(\mathbb{R}^2)}$ but it controls $||\varepsilon||_{L^4_{loc}(\mathbb{R}^2)}$.

Decomposition with a cutoff

For $R > 0$ we define χ_R a cutoff with $\chi_R = 1$ in $B(0, R)$ and 0 outside of $B(0, 2R)$.

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For $R > 0$ we define χ_R a cutoff with $\chi_R = 1$ in $B(0, R)$ and 0 outside of $B(0, 2R)$. We decompose

$$\mathcal{E}(V_1 + \varepsilon) = B_R(\varepsilon) + N_R(\varepsilon)$$

with

$$B_R(\varepsilon) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla \varepsilon|^2 - (1 - |V_1|^2)|\varepsilon|^2 + 2\chi_R^2 \mathfrak{Re}^2(\overline{V_1}\varepsilon)$$

and

$$N_R(\varepsilon) := \frac{1}{4} \int_{\mathbb{R}^2} (1 - \chi_R^2)(2\mathfrak{Re}(\overline{V_1}\varepsilon) + |\varepsilon|^2)^2 + \chi_R^2(4\mathfrak{Re}(\overline{V_1}\varepsilon)|\varepsilon|^2 + |\varepsilon|^4).$$

Still not coercive

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Still not coercive

The quadratic form B_R has infinitely many negative eigenvalues. This is because of the term $\mathcal{P}_{\mathcal{R}}(\varepsilon) := P_R(V_1 + \varepsilon)$ where

$$P_R(\psi) := \int_{\mathbb{R}^2} (1 - \chi_R)^2 \frac{x^\perp}{|x|^2} . \mathfrak{Re}(i\overline{\psi} V_1 \nabla(\psi \overline{V_1})).$$

We decompose

$$\mathcal{E}(V_1 + \varepsilon) = \mathcal{B}_R(\varepsilon) + \mathcal{P}_{\mathcal{R}}(\varepsilon) + N_R(\varepsilon).$$

The final decomposition

1) The renormalized energy $\mathcal{E}$ is conserved by the flow
2) $\mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) = \mathcal{B}_R(\varepsilon) + \mathcal{P}_{\mathcal{R}}(\varepsilon) + N_R(\varepsilon)$
3) $\mathcal{B}_R(\varepsilon) \geqslant \kappa||\varepsilon||_?^2$ under three orthogonality conditions
4) $N_R(\varepsilon) + \mathcal{P}_{\mathcal{R}}(\varepsilon) \geqslant -K||\varepsilon||_?^3$
5) $1 \gg \mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) \geqslant \kappa||\varepsilon||_?^2(1 - K||\varepsilon||_?)$

Coercivity of $\mathcal{B}_R$

Proposition

There exist universal constants $\kappa_0 > 0$ and $N_0 > 0$ such that, given any $\varepsilon \in H$ verifying the orthogonality conditions

$$\int_{\mathbb{R}^2} \chi \mathfrak{Re}(\overline{\varepsilon} \partial_{x_1} V_1) = \int_{\mathbb{R}^2} \chi \mathfrak{Re}(\overline{\varepsilon} \partial_{x_2} V_1) = \int_{\mathbb{R}^2} \chi \mathfrak{Re}(\overline{\varepsilon} i V_1) = 0,$$

and for any $R_0 \geqslant 1$, there exists $R_0 \leqslant R \leqslant 2^{N_0} R_0$ such that

$$\mathcal{B}_R(\varepsilon) \geqslant \kappa_0 (||\varepsilon||_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \mathfrak{Re}^2(\overline{\varepsilon} V_1)).$$

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Sketch of the proof

- We decompose ε in harmonics
- Locally the analysis is similar to Del Pino, Felmer, Kowalczyk (working with $\chi_R\varepsilon$)
- We have removed the non controllable part $\mathcal{P}_{\mathcal{R}}(\varepsilon)$ at infinity
- The cutoff location R is chosen to glue these two estimates together

The key difficulty is to have

$$\int_{\mathbb{R}^2} |\nabla(\chi_R \overline{V_1}\varepsilon)|^2 + |\nabla((1-\chi_R)\overline{V_1}\varepsilon)|^2 \geqslant \kappa \int_{\mathbb{R}^2} |\nabla(\overline{V_1}\varepsilon)|.$$

On the right path

1) The renormalized energy $\mathcal{E}$ is conserved by the flow
2) $\mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) = \mathcal{B}_R(\varepsilon) + \mathcal{P}_{\mathcal{R}}(\varepsilon) + N_R(\varepsilon)$
3) $\mathcal{B}_R(\varepsilon) \geqslant \kappa_0(||\varepsilon||_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \mathfrak{Re}^2(\overline{\varepsilon} V_1)))$ under three orthogonality conditions
4) $N_R(\varepsilon) + \mathcal{P}_{\mathcal{R}}(\varepsilon) \geqslant -K||\varepsilon||_?^3$
5) $1 \gg \mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) \geqslant \kappa||\varepsilon||_?^2(1 - K||\varepsilon||_?)$

The two "nonlinear" terms

We estimate

$$N_R(\varepsilon) \geqslant \frac{\kappa_0}{8}||\eta_\varepsilon||^2_{L^2(\mathbb{R}^2)} - ||\varepsilon||^3_{L^3(B_{2R})} - \frac{\kappa_0}{2}\int_{\mathbb{R}^2}\chi_R^2 \mathfrak{Re}^2(\overline{\varepsilon} V_1)$$

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Lemma

There exist universal constant $\delta > 0, \Lambda > 0$ and $K \geqslant 1$ such that, for $V_1 + \varepsilon \in E$, if $R \geqslant \Lambda$ and $||\varepsilon||_H + ||\eta_\varepsilon||_{L^2} \leqslant \delta$, then

$$|\mathcal{P}_\mathcal{R}(\varepsilon)| \leqslant \frac{K}{R}(||\varepsilon||^2_H + ||\eta_\varepsilon||^2_{L^2}).$$

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$$|\mathcal{P}_\mathcal{R}(\varepsilon)| \leqslant \frac{K}{R}(||\varepsilon||^2_H + ||\eta_\varepsilon||^2_{L^2}).$$

This uses a decomposition in frequencies of ε first introduced by Patrick Gérard.

Final estimations

We choose R_0 large enough such that $\frac{K}{R} \leqslant \frac{\kappa_0}{16}$

1) The renormalized energy $\mathcal{E}$ is conserved by the flow
2) $\mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) = \mathcal{B}_R(\varepsilon) + \mathcal{P}_{\mathcal{R}}(\varepsilon) + N_R(\varepsilon)$
3) $\mathcal{B}_R(\varepsilon) \geqslant \kappa_0(||\varepsilon||_H^2 + \int_{\mathbb{R}^2} \chi_R^2 \mathfrak{Re}^2(\overline{\varepsilon} V_1)))$ under three orthogonality conditions
4) $N_R(\varepsilon) + \mathcal{P}_{\mathcal{R}}(\varepsilon) \geqslant \frac{\kappa_0}{16}(||\eta_\varepsilon||_{L^2}^2 - ||\varepsilon||_H^2) - K||\varepsilon||_H^3$
5) $1 \gg \mathcal{E}(V_1(. - X(t))e^{i\gamma(t)} + \varepsilon) \geqslant \frac{\kappa_0}{16}(||\varepsilon||_H^2 + ||\eta_\varepsilon||_{L^2}^2) - K||\varepsilon||_H^3$

Main theorem

We recall

$$E = \{\psi \in H^1_{loc}(\mathbb{R}^2), ||\psi||_H < +\infty, 1 - |\psi|^2 \in L^2(\mathbb{R}^2)\}$$

for $||\psi||_H^2 = \int_{\mathbb{R}^2} |\nabla(\overline{V_1}\psi)|^2 + \frac{|\nabla\psi|^2}{1+r^2}$. We consider the distance

$$d_E(\psi_1, \psi_2) := ||\psi_1 - \psi_2||_H + |||\psi_1|^2 - |\psi_2|^2||_{L^2}.$$

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Theorem

There exist universal constant $\delta > 0, C > 0$ such that, if $\psi_0 \in E$ and $d_E(V_1, \psi_0) \leqslant \delta$, then the solution ψ_t with initial data ψ_0 of the Gross-Pitaevskii equation satisfies

$$d_E(V_1, Orb(\psi_t)) \leqslant C d_E(V_1, \psi_0)$$

for any $t \in \mathbb{R}$, where $Orb(\psi) := \{e^{i\gamma}\psi(. + X), \gamma \in \mathbb{R}, X \in \mathbb{R}^2\}$.

Control of the parameters

Proposition

There exist universal constants $\tau > 0, C > 0$ such that, if $\psi_0 \in E$ and $d_E(V_1, \psi_0) \leqslant \tau$, then there exists two functions $X \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^2)$ and $\gamma \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$ such that

$$d_E(V_1, e^{i\gamma(t)}\psi_t(. + X(t))) \leqslant C d_E(V_1, \psi_0)$$

and

$$|X'(t)| + |\gamma'(t)| \leqslant C d_E(V_1, \psi_0).$$

Open problems

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- Orbital stability for a pair of vortices

Pair of vortices

Theorem (Chiron, P.)

There exists $c_0 > 0$ a small constant such that, for any $0 < c \leqslant c_0$, there exists a solution of $(\mathrm{TW}_c)(u) = ic\partial_{x_2} u + \Delta u - (|u|^2 - 1)u = 0$ of the form

$$Q_c = V_1(. - d_c \overrightarrow{e_1}) V_{-1}(. + d_c \overrightarrow{e_1}) + \Gamma_{c,d_c},$$

where $d_c = \frac{1 + o_{c \to 0}(1)}{c}$ is a continuous function of c and $\|\Gamma_{c,d_c}\|_{C^1(\mathbb{R}^2,\mathbb{C})} = o_{c \to 0}(1)$.

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Theorem (Chiron, P.)

The function Q_c is, for c small enough, the unique (up to translation and shift of phase) minimizer of the energy at fixed momentum. As such, it is orbitally stable for the semi distance

$$D_0(u,v) = ||\nabla u - \nabla v||_{L^2} + |||u| - |v|||_{L^2}.$$

Done

Thank you for your attention !