Quasilinear Maxwell equations

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Maxwell's equations in vacuum

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- Electric charges are the sources of the electric field,
- Magnetic fields have no sources,
- Circulating electric field generates a magnetic field,
- Currents make magnetic fields.

Maxwell's equations from SRT and gauge theory

Starting from Lorentz force: build a theory, which complies with special relativity: $SO(1,3)$. u^μ: 4-velocity, $\tau = (dx^\mu dx_\mu)^{1/2}$-proper time:
$u'^\mu = \Lambda^\mu_{\cdot\nu} u^\nu, \quad u^\mu u_\mu = (u^0)^2 - \sum_{i=1}^3 (u^i)^2 = (u')^\mu (u')_\mu$.

$$\frac{du^\mu}{d\tau} = F^{\mu\nu} u_\nu, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \quad A^\mu \to A^\mu + \partial^\mu \chi.$$

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Lagrange density from Lorentz scalars:

$$\mathcal{L}[A^\mu] = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + j^\mu A_\mu, \quad S[A^\mu] = \int \mathcal{L} d^4 x.$$

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$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -B_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}, \quad j^\mu = \begin{pmatrix} \rho \\ j^1 \\ j^2 \\ j^3 \end{pmatrix}.$$

Maxwell's equations in media

Electric fields applied to material: alignment of charges and flexibility

$\rightarrow$ bound charges: P ... polarization,

$\rightarrow$ bound currents: M ... magnetization.

- displacement and magnetizing field: $D = E + P$, $H = B - M$.

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Constitutive equations:

$D(t,x) = \varepsilon(t,x)E(t,x), \quad B(t,x) = \mu(t,x)H(t,x).$

ε, μ: permittivity / permeability, symmetric and positive definite.

Maxwell's equations in 2d

fields and permittivity depend only on t, x_1, x_2.

$$D_3 = E_3 = 0, \quad H_i = B_i = 0, \; i \in \{1,2\}; \qquad \varepsilon = \begin{pmatrix} \varepsilon^{11} & \varepsilon^{12} \\ \varepsilon^{21} & \varepsilon^{22} \end{pmatrix}, \; \mu = 1.$$

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$$\begin{cases} \nabla \cdot D &= \rho, \\ \partial_t D &= \nabla_\perp H + j, \\ \partial_t B &= -\nabla \times E. \end{cases}$$

$$\nabla_\perp = (\partial_2, -\partial_1); \quad (D, E) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2, \; (H, B) : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}.$$

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$$\begin{cases} \partial_t D &= \nabla_\perp H, \\ \partial_t B &= -\nabla \times E. \end{cases} \qquad P(x,\partial) \begin{pmatrix} D \\ H \end{pmatrix} = F.$$

$$P(x,\partial) = \begin{pmatrix} \partial_t & 0 & -\partial_2 \\ 0 & \partial_t & \partial_1 \\ -\partial_2(\varepsilon_{11}\cdot) + \partial_1(\varepsilon_{21}\cdot) & \partial_1(\varepsilon_{22}\cdot) - \partial_2(\varepsilon_{12}\cdot) & \partial_t \end{pmatrix}.$$

Kerr nonlinearity

$\varepsilon = \varepsilon(D), \quad \varepsilon = (1 + |E|^2) 1_{2\times 2}$. Aim: Solve

$$\begin{cases} P(x,\partial)(D,H) &= 0, \\ (D,H)(0) &= (D_0, H_0) \in H^s(\mathbb{R}^2;\mathbb{R}^3), \\ \nabla \cdot D_0 &= 0 \end{cases} \qquad (1)$$

with Kerr nonlinearity for rough initial data.

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- Spitz (2019): Boundary conditions etc.
- here: full space, take advantage of dispersion.

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$$\partial_t^2 E = \Delta E, \quad \partial_t^2 B = \Delta B.$$

Sharp Strichartz estimates for wave equation due to Keel–Tao (1998).

$$u(t,x) = \cos(t|D'|)u_0(x) + \frac{\sin(t|D'|)}{|D'|}u_1(x) :$$

$$\| |D'|^{1-\rho} u \|_{L^p(\mathbb{R};L^q(\mathbb{R}^d))} \lesssim \|u_0\|_{\dot{H}^1(\mathbb{R}^d)} + \|u_1\|_{L^2(\mathbb{R}^d)}$$

$$p, q \geq 2,\ \tfrac{2}{p} + \tfrac{d-1}{q} \leq \tfrac{d-1}{2},\ (p,q,d) \neq (2,\infty,3),\ \rho = d\left(\tfrac{1}{2} - \tfrac{1}{q}\right) - \tfrac{1}{p}.$$

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$$\partial_t H = \partial_2(\varepsilon_{11} D_1) - \partial_1(\varepsilon_{12} D_2) - \partial_2(\varepsilon_{12} D_1) - \partial_1(\varepsilon_{22} D_2) \quad (\varepsilon = \varepsilon(x)).$$

$$\partial_t^2 H = \partial_2(\varepsilon_{11}\partial_2 H) - \partial_1(\varepsilon_{21}\partial_2 H) - \partial_2(\varepsilon_{12}\partial_1)H + \partial_1(\varepsilon_{22}\partial_1)H =: \Delta_{\varepsilon^{-1}} H.$$

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$$\partial_t^2 H = \sum_{i,j=1}^{d} \partial_i(g^{ij}\partial_j H), \quad g^{ij}(t,x): \text{ uniformly elliptic.}$$

- (1989) Kapitanskii: C^∞-coefficients; (1991) Mockenhaupt–Seeger–Sogge
- (1998) Smith: $C^{1,1}$; (1999) Bahouri–Chemin, Tataru C^s-coefficients $s < 2$.
- (2001) Tataru, (2002) Smith–Tataru: full range of C^s coefficients with sharp derivative loss

Ψdos and FBI transform

$a \in S^m_{\rho,\delta}$: $a \in C^\infty(\mathbb{R}^{2d})$, $|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| \lesssim_{\alpha,\beta} (1+|\xi|)^{m-\rho|\beta|+|\alpha|\delta}$.

$$a(x,\partial)u = C_d \int_{\mathbb{R}^d} e^{ix.\xi} a(x,\xi) \hat{u}(\xi) d\xi.$$

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$$T_\lambda : L^2(\mathbb{R}^d) \to L^2_\Phi(T^* R^d) \simeq L^2_\Phi(R^{2d}), \quad T_\lambda(a_\lambda(y,\partial)u_\lambda) \approx \tilde{a}_\lambda(y,\xi) T_\lambda u_\lambda.$$

$$T_\lambda f(z) = \lambda^{\frac{3d}{4}} 2^{-\frac{d}{2}} \pi^{-\frac{3d}{4}} \int_{\mathbb{R}^d} e^{-\frac{\lambda}{2}(z-y)^2} f(y) dy, \quad z = x - i\xi \in \mathbb{C}^d.$$

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$$f_{x_0,\xi_0}(y) = \lambda^{\frac{d}{4}} \pi^{-\frac{d}{4}} e^{-\frac{\lambda}{2}(y-x_0)^2} e^{i\lambda(y-x_0)\xi_0}, \quad \Phi(z) = e^{-\lambda|\Im z|^2};$$

$$T_\lambda f_{x_0,\xi_0}(z) = \lambda^{-\frac{d}{4}} \pi^{\frac{d}{4}} e^{-\frac{\lambda}{4}|z-(x_0-i\xi_0)|^2} e^{\frac{\lambda}{2}|\Im z|^2} e^{-i\lambda(\Re z - x_0)(\Im z - \xi_0)}.$$

Conjugation

$$a(x,\xi) = 0,\ \xi \notin B(0,2) \backslash B(0,1/2), \quad a_\lambda(x,\xi) = a(x,\xi/\lambda).$$

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$$T_\lambda(yf)(z) = (x - \frac{1}{i\lambda}(\partial_\xi - \lambda\xi)) T_\lambda f$$

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$$T_\lambda a_\lambda(x,\partial) \approx \sum_{\alpha,\beta}(\partial_\xi - \lambda\xi)^\alpha \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{|\alpha|!|\beta|!(-i\lambda)^{|\alpha|}\lambda^{|\beta|}}\left(\frac{1}{i}\partial_x - \lambda\xi\right)^\beta T_\lambda.$$

$$\tilde{a}^s_\lambda = \sum_{|\alpha|+|\beta|<s}(\partial_\xi - \lambda\xi)^\alpha \frac{\partial_x^\alpha \partial_\xi^\beta a(x,\xi)}{|\alpha|!|\beta|!(-i\lambda)^{|\alpha|}\lambda^{|\beta|}}\left(\frac{1}{i}\partial_x - \lambda\xi\right)^\beta.$$

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$$s\leq 1:\ \tilde{a}_\lambda^s=a,\quad 1<s\leq 2:\tilde{a}_\lambda^s=a-\frac{1}{i\lambda}a_x(\partial_\xi-\lambda\xi)+\frac{1}{\lambda}a_\xi(\frac{1}{i}\partial_x-\lambda\xi).$$

Approximation of rough symbols

Symbols $a(x,\xi) \in C_x^s C_c^\infty$: $a(x,\xi) = 0$ if $\xi \notin B(0,2) \backslash B(0,1/2)$.

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Approximation theorem (Tataru (1999)):

$$a \in C^s_x C^\infty_c : \quad \| R^s_{\lambda,a} \|_{L^2 \to L^2_\Phi} \lesssim \lambda^{-\frac{s}{2}}.$$

Fourier series trick:

$$a(x,\xi) = a(x,\xi)\beta(\xi) = \sum_{k \in \mathbb{Z}^d} e^{ik.\xi} \hat{\tilde{a}}(x,k), \quad \hat{\tilde{a}}(x,k) = \int_{\mathbb{T}^d} e^{-i\xi.k} \tilde{a}(x,\xi) d\xi.$$

Separates variables: $a(x,\xi) \approx b(x)c(\xi)$.

Strichartz estimates for C^1-coefficients

$$\| |D|^{1-\rho-\frac{1}{4}} u \|_{L^p L^q} \lesssim (1 + \|g\|_{C^1}) \|u\|_{H^1} + \|Qu\|_{H^{\frac{-1}{2}}}.$$

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1. Reduce the identity $Q(x,\partial)u = \partial_t^2 - \partial_i(g^{ij}\partial_j)u = f$ to dyadic frequency blocks (modulo lower order terms):

$$Q_\lambda(x,\partial)u_\lambda = f_\lambda.$$

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2. Apply the FBI transform and use the approximation result:

$$T_\lambda(Q_\lambda(x,\partial)u_\lambda) = T_\lambda f_\lambda \Rightarrow q(x,\xi)T_\lambda u_\lambda = T_\lambda f_\lambda + O(\lambda^{-1/2}).$$

Main contribution:

$$T_\lambda u_\lambda = \frac{T_\lambda f_\lambda}{q(x,\xi)} \Rightarrow u_\lambda = T_\lambda^* \frac{T_\lambda f_\lambda}{q(x,\xi)}.$$

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3. Use oscillatory integral estimates to prove Strichartz estimates.

Strichartz estimates for 2d Maxwell

$u = (D_1, D_2, B)$, $\partial_1 D_1 + \partial_2 D_2 = \rho_e$:

$$P(x,\partial) = \begin{pmatrix} \partial_t & 0 & -\partial_2 \\ 0 & \partial_t & \partial_1 \\ -\partial_2(\varepsilon_{11}\cdot) + \partial_1(\varepsilon_{12}\cdot) & -\partial_2(\varepsilon_{21}\cdot) + \partial_1(\varepsilon_{22}\cdot) & \partial_t \end{pmatrix}.$$

Applying FBI transform:

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$p(x,\xi) \in \mathbb{C}^{3\times 3}$!

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$$T_\lambda(P_\lambda(x,\partial)u_\lambda) = T_\lambda f_\lambda \Rightarrow p(x,\xi)T_\lambda u_\lambda = T_\lambda f_\lambda.$$

$p(x,\xi) \in \mathbb{C}^{3\times 3}$! Remedy: $p(x,\xi) = m(x,\xi)d(x,\xi)m^{-1}(x,\xi)$.

$$d(x,\xi) = i \begin{pmatrix} \xi_0 & 0 & 0 \\ 0 & \xi_0 + \|\xi'\|_{\tilde{\varepsilon}} & 0 \\ 0 & 0 & \xi_0 - \|\xi'\|_{\tilde{\varepsilon}} \end{pmatrix}, \quad \|\xi'\|_{\tilde{\varepsilon}} = \Big(\sum_{i,j=1}^{2} \tilde{\varepsilon}^{ij}\xi_i'\xi_j'\Big)^{1/2}.$$

$m(x,\xi)$, $m^{-1}(x,\xi)$: symbols of Riesz transforms, L^p-bounded.

Diagonalization: $m(x,\xi)d(x,\xi)m^{-1}(x,\xi) = p(x,\xi)$.

$$m^{-1}(x,\xi) = \begin{pmatrix} -\xi_1^* & -\xi_2^* & 0 \\ \frac{\xi_2^* \varepsilon_{11}(x) - \xi_1^* \varepsilon_{12}(x)}{2} & \frac{-\xi_1^* \varepsilon_{22}(x) + \xi_2^* \varepsilon_{12}(x)}{2} & \frac{1}{2} \\ \frac{-\xi_2^* \varepsilon_{11}(x) + \xi_1^* \varepsilon_{12}(x)}{2} & \frac{\xi_1^* \varepsilon_{22}(x) - \xi_2^* \varepsilon_{12}(x)}{2} & \frac{1}{2} \end{pmatrix}, \quad \xi_i^* = \frac{\xi_i}{\|\xi\|_{\tilde{\varepsilon}}}.$$

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Find ΨDOs: $\mathcal{M}\mathcal{D}\mathcal{N} = P + E, \quad \|E\|_{L^2\to L^2} \lesssim 1$.

$$\mathcal{D}(x,\partial) = \mathrm{diag}(\partial_t, \partial_t - iD_{\tilde{\varepsilon}}, \partial_t + iD_{\tilde{\varepsilon}}), \quad D_{\tilde{\varepsilon}} = Op(\sqrt{\tilde{\varepsilon}^{ij}\xi_i\xi_j}).$$

$$\mathcal{M}(x,\partial) = \begin{pmatrix} \frac{i}{D_{\tilde{\varepsilon}}}(\partial_1(\varepsilon_{22}\cdot) - \partial_2(\varepsilon_{12}\cdot)) & \frac{-i}{D_{\tilde{\varepsilon}}}\partial_2 & \frac{i}{D_{\tilde{\varepsilon}}}\partial_2 \\ \frac{i}{D_{\tilde{\varepsilon}}}(\partial_2(\varepsilon_{11}\cdot) - \partial_1(\varepsilon_{12}\cdot)) & \frac{i}{D_{\tilde{\varepsilon}}}\partial_1 & \frac{-i}{D_{\tilde{\varepsilon}}}\partial_1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{N}(x,\partial) = \begin{pmatrix} i\partial_1 \frac{1}{D_{\tilde{\varepsilon}}} & i\partial_2 \frac{1}{D_{\tilde{\varepsilon}}} & 0 \\ \frac{i(\varepsilon_{12}\partial_1 - \varepsilon_{11}\partial_2)}{2}\frac{1}{D_{\tilde{\varepsilon}}} & \frac{i(\varepsilon_{22}\partial_1 - \varepsilon_{12}\partial_2)}{2}\frac{1}{D_{\tilde{\varepsilon}}} & \frac{1}{2} \\ \frac{i(\varepsilon_{11}\partial_2 - \varepsilon_{12}\partial_1)}{2}\frac{1}{D_{\tilde{\varepsilon}}} & \frac{i(\varepsilon_{12}\partial_2 - \varepsilon_{22}\partial_1}{2}\frac{1}{D_{\tilde{\varepsilon}}} & \frac{1}{2} \end{pmatrix}.$$

Theorem (rsc-Schnaubelt)

Let $\varepsilon^{ij} \in C^2$ *uniformly elliptic. Let* $u = (D_1, D_2, H)$ *with* $\partial_1 D_1 + \partial_2 D_2 = \rho_e$. *Then,*

$$\| |D|^{-\rho} u \|_{L_t^p L_x^q} \lesssim \alpha \| u \|_{L^2_{t,x}} + \alpha^{-1} \| P u \|_{L^2_{t,x}} + \| |D|^{-\frac{1}{2}} \rho_e \|_{L^2_{t,x}}$$

with $\| \partial^2 \varepsilon \|_{L^\infty} \le \alpha^4$, $|D| = \mathcal{F}^{-1}(|(\tau, \xi')| \mathcal{F})$.

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with $\|\partial^2 \varepsilon\|_{L^\infty} \leq \alpha^4$, $|D| = \mathcal{F}^{-1}(|(\tau, \xi')|\mathcal{F})$.

Remarks:

- Strichartz estimates for C^s-coefficients, $0 < s < 2$ as a corollary,
- estimates are sharp for C^s-coefficients, $1 \leq s < 2$ by testing with example,
- estimates for C^2-coefficients remain true provided that $\|\partial^2 \varepsilon\|_{L^1 L^\infty} < \infty$.

Strichartz estimates for quasilinear equations

Corollary

Assume $\|\partial_x \varepsilon\|_{L^2 L^\infty} \lesssim 1$, $\partial_1 u_1 + \partial_2 u_2 = 0$. *Let* $(\rho, p, q, 2)$ *Strichartz pair. Then,*

$$\|\langle D' \rangle^{-\alpha} u\|_{L^p(0,T;L^q)} \lesssim_T \|u_0\|_{L^2} + \|Pu\|_{L^1(0,T;L^2)}$$

for $\alpha > \rho + \frac{1}{3p}$.

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Rewrite Kerr nonlinearity

$$\begin{cases} \partial_t u_1 &= \partial_2 u_3, \quad u(0) = u_0 \in H^s(\mathbb{R}^2;\mathbb{R})^3, \\ \partial_t u_2 &= -\partial_1 u_3, \qquad \partial_1 u_1 + \partial_2 u_2 = 0, \\ \partial_t u_3 &= \partial_2(\varepsilon^{-1}(u) u_1) - \partial_1(\varepsilon^{-1}(u) u_2), \end{cases}$$

where $\varepsilon^{-1}(u) = \psi(|u_1|^2 + |u_2|^2)$, $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 1}$ smooth, monotone increasing.

Local well-posedness for quasilinear equations

- Energy estimates: $E^s(u(t)) \lesssim e^{c(A) \int_0^t B(t') dt'} E^s(u(0))$, $E^s \approx_A \|u\|_{H^s}$, $A = \sup_{0 \leq t' \leq t} \|u(t')\|_{L^\infty}$, $B(t) = \|\nabla_{x'} u(t)\|_{L^\infty}$.
- L^2-Lipschitz bounds for differences: $v = u^1 - u^2$, $\|v(t)\|_{L^2}^2 \lesssim e^{c(A) \int_0^t B(t') dt'} \|v(0)\|_{L^2}^2$.
- Frequency envelopes (cf. Tao (2001)).

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$$\|u\|_{E^s}^2 = \langle \langle D' \rangle^s u, C(u) \langle D' \rangle^s u \rangle \approx_A \|u\|_{H^s}^2.$$

Rewrite

$$\partial_t u = \mathcal{A}^j(u) \partial_j u, \quad (\mathcal{A}^j(u))^* C(u) = C(u) \mathcal{A}^j(u).$$

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$$\|u\|_{E^s}^2 \lesssim_A B(t) \|u(t)\|_{E^s}^2.$$

Strichartz estimates: $\|\nabla_{x'} u\|_{L^4(0,T;L^\infty)} \lesssim \|u_0\|_{H^s}$ give a priori energy estimates for $s > \frac{11}{6}$.

Improved local well-posedness

2d Maxwell equations:

$$\begin{cases} \partial_t D & = \nabla_\perp H, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2, \\ \partial_t B & = -\nabla \times E, \quad \nabla \cdot D = 0, \\ (D(0), B(0)) & \in H^s(\mathbb{R}^2)^3. \end{cases} \qquad (2)$$

$H = B$, $D = (1 + |E|^2)E$.

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Theorem (rsc-Schnaubelt)

(2) with Kerr nonlinearity is locally well-posed for $s > 11/6$.

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3d Maxwell equations:

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Theorem (rsc)

(3) *with Kerr nonlinearity is locally well-posed for* $s > 13/6$.

Time-harmonic Maxwell equations

monochromatic ansatz:

$$\mathcal{D}(t,x) = e^{i\omega t} D(x),\ \mathcal{E}(t,x) = e^{i\omega t} E(x), \dots$$

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constant-coefficient permittivity / permeability: $\varepsilon = \text{diag}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\mu = 1_{3\times 3}$.

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$$P(\omega, D) = \begin{pmatrix} i\omega 1_{3\times 3} & -\nabla \times \\ \nabla \times (\varepsilon^{-1} \cdot) & i\omega 1_{3\times 3}. \end{pmatrix}, \quad P(\omega, D)(D, H) = (J_e, J_m).$$

Aim: Solve for $\omega \in \mathbb{R} \backslash 0$:

$$(D, H) = P(\omega, D)^{-1} (J_e, J_m).$$

Problem:

$$(P(\omega, D)^{-1} f)_i \widehat{(\xi)} \sim \frac{\hat{f}(\xi)}{\|\xi\| - \omega}.$$

ill-defined in the distributional sense.

Regularization: $\omega \in \mathbb{C} \backslash \mathbb{R}$. Prove estimates

$$\|(D,B)\|_{L_0^q} = \|P(\omega,D)^{-1}(J_e,J_m)\|_{L_0^q} \lesssim_{\omega,p,q} \|(J_e,J_m)\|_{L_0^p}$$

which are bounded in ω for ω in a compact set away from the origin and take limits $\Im\omega \uparrow 0$ and $\Im\omega \downarrow 0$.

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$\varepsilon = \text{diag}(a^{-1}, b^{-1}, b^{-1})$ (partially anisotropic case).

$$\begin{aligned} \|\xi\|^2 &= \xi_1^2 + \xi_2^2 + \xi_3^2, \quad & \|\xi\|_\varepsilon^2 &= b\xi_1^2 + a\xi_2^2 + a\xi_3^2, \\ \xi' &= \xi/\|\xi\|, & \tilde{\xi} &= \xi/\|\xi\|_\varepsilon. \end{aligned}$$

$$(\nabla \times u)\widehat{\;}(\xi) = -i\mathcal{B}(\xi)\hat{u}(\xi), \quad \mathcal{B}(\xi) = \begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix}.$$

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Theorem (rsc)

$1 < p,q < \infty$ and $\omega \in \mathbb{C}\backslash\mathbb{R}$.

$$\|P(\omega,D)^{-1}\|_{L_0^p \to L_0^q} \sim \|((-\Delta)^{1/2} - \omega)^{-1}\|_{p\to q} + \|((-\Delta)^{1/2} + \omega)^{-1}\|_{p\to q}.$$

Limiting operators

Sokhotsky's formula:

$$\frac{1}{x \pm i\varepsilon} \to \text{p.v.}\frac{1}{x} \mp i\pi\delta_0 \text{ for } \varepsilon \downarrow 0.$$

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Local limiting operators:

$$(\mathcal{R}^{\text{loc}}_{\delta,\pm} f)(x) = \int_{\mathbb{R}^3} \frac{e^{ix.\xi}\hat{f}(\xi)}{\|\xi\| - (\omega \pm i\delta)} d\xi$$

$$\to \text{p.v.} \int \frac{e^{ix.\xi}\hat{f}(\xi)}{\|\xi\| - \omega} d\xi \pm i\pi \int_{\{\|\xi\|=\omega\}} e^{ix.\xi}\hat{f}(\xi) d\sigma(\xi).$$

Restriction–extension operator:

$$(\mathcal{R}^{\text{loc}}_{+} f)(x) - (\mathcal{R}^{\text{loc}}_{-} f)(x) = 2\pi i \int_{\{\|\xi\|=\omega\}} e^{ix.\xi}\hat{f}(\xi) d\sigma(\xi).$$

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This is a special case of the Bochner–Riesz operator with negative index:

$$(\mathcal{B}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_{\mathbb{R}^d} e^{ix.\xi}(1 - \|\xi\|)_+^{-\alpha}\hat{f}(\xi) d\xi.$$

Fully anisotropic case: $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3 \neq \varepsilon_1$, $\mu = 1_{3\times 3}$. (joint work with R. Mandel)

$$\begin{cases} i\omega D &= \nabla \times H - J_e, \quad \nabla \cdot J_e = \nabla \cdot J_m = 0, \\ i\omega B &= -\nabla \times E + J_m. \end{cases} \tag{4}$$

- attempt to diagonalize leads to singular Fourier multipliers.

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Lemma

$(E, H) \in (\mathcal{S}'(\mathbb{R}^3))^6$ *solve* (4), *then*

$$\begin{cases} (M_E(\xi) - \omega^2)\hat{E}(\xi) &= -i\omega\varepsilon^{-1}\hat{J}_e + i\varepsilon^{-1}\mathcal{B}(\xi)\mu^{-1}\hat{J}_m, \\ (M_H(\xi) - \omega^2)\hat{H}(\xi) &= i\mu^{-1}\mathcal{B}(\xi)\varepsilon^{-1}\hat{J}_e(\xi) + i\omega\mu^{-1}\hat{J}_m(\xi) \end{cases}$$

with

$$M_E(\xi) = -\varepsilon^{-1}\mathcal{B}(\xi)\mu^{-1}\mathcal{B}(\xi), \quad M_H(\xi) = -\mu^{-1}\mathcal{B}(\xi)\varepsilon^{-1}\mathcal{B}(\xi).$$

$p(\omega, \xi) := \det(M_E(\xi) - \omega^2)$ characteristic surface.

For low frequencies $p(\omega, \xi)$ can vanish. Define $\mathcal{N}(\eta) = 1 - q_0^*(\eta) + q_1^*(\eta)$ with

$$q_0^*(\eta) = \eta_1^2\left(\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right) + \eta_2^2\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_3}\right) + \eta_3^2\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right),$$

$$q_1^*(\eta) = \frac{1}{\varepsilon_1 \varepsilon_2 \varepsilon_3}(\varepsilon_1 \eta_1^2 + \varepsilon_2 \eta_2^2 + \varepsilon_3 \eta_3^2)(\eta_1^2 + \eta_2^2 + \eta_3^2).$$

$S^* = \{\eta \in \mathbb{R}^3 : \mathcal{N}(\eta) = 0\}$: Fresnel surface.

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$S^* = \{\eta \in \mathbb{R}^3 : \mathcal{N}(\eta) = 0\}$: Fresnel surface.

Theorem (Darboux)

S^* admits a decomposition $S^* = S_1 \cup S_2 \cup S_3$:

(i) *S_1 smooth regular manifold with two non-vanishing principal curvatures,*

(ii) *S_2 smooth regular manifold with one non-vanishing principal curvature,*

(iii) *S_3 neighbourhoods of conic points.*

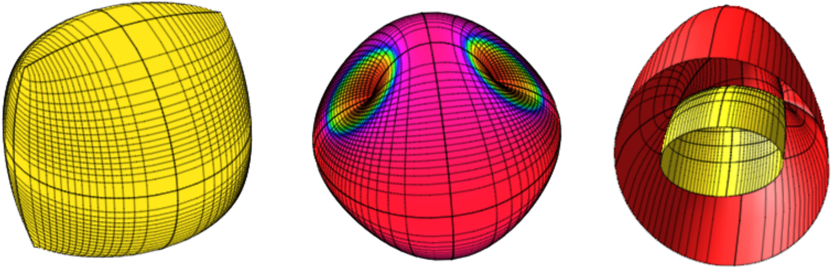

Figure: Fresnel's wave surface: inner sheet (left) and outer sheet (middle) for $\varepsilon_1 = 1$, $\varepsilon_2 = 3$, $\varepsilon_3 = 9$. The colours on the outer sheet highlight regions of identical Gaussian curvature. The blue Hamiltonian circles encase the singular points. The contact of inner (yellow) and half of the outer sheet (red) at two singular points is depicted in the right figure.

Generalized Bochner–Riesz estimates: Let $\xi = (\xi', \xi_d) \in \mathbb{R}^{d-1} \times \mathbb{R}^d$.

$$(T^\alpha f)\widehat{(\xi)} = \frac{(\xi_d - \psi(\xi'))_+^{-\alpha}}{\Gamma(1-\alpha)} \chi(\xi') \hat{f}(\xi), \quad \chi \in C_c^\infty([-1,1]^{d-1}), \quad 0 < \alpha < \frac{k+2}{2}.$$

$S = \{(\xi', \psi(\xi')) : \xi' \in [-1,1]^{d-1}\}$ be a smooth surface with $k \in \{1, \ldots, d-1\}$ principal curvatures bounded from below.

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Ingredients in the proof:

- Convenient decomposition of distribution D^α,
- Dispersive estimate,
- Kernel estimate $T_\delta f = K_\delta * f$,
- Bourgain's summation lemma.

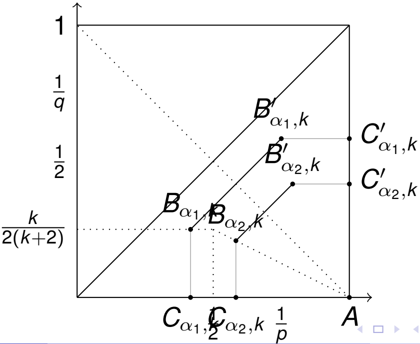
Theorem (Mandel–rsc)

Let $1 \leq p, q \leq \infty$, $d \in \mathbb{N}$, $d \geq 3$, *and* $0 < \alpha < \frac{k+2}{2}$. *Then*

$$\|T^\alpha f\|_{L^q} \lesssim_{\alpha,p,q} \|f\|_{L^p}$$

holds true for $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{P}_\alpha(k)$ *with*

$$\mathcal{P}_\alpha(k) = \left\{ (x,y) \in [0,1]^2 : \, x > \frac{k+2\alpha}{2(k+1)}, \, y < \frac{k+2-2\alpha}{2(k+1)}, \, x - y \geq \frac{2\alpha}{k+2} \right\}.$$



Theorem (Mandel–rsc)

Let $1 \leq p_1, p_2, q \leq \infty$, $\varepsilon = diag(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\mu = diag(\mu_1, \mu_2, \mu_3)$ *satisfy* $\varepsilon_1/\mu_1 \neq \varepsilon_2/\mu_2 \neq \varepsilon_3/\mu_3 \neq \varepsilon_1/\mu_1$ *and* $(J_e, J_m) \in L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)$ *divergence-free. If*

$$\frac{1}{p_1} > \frac{3}{4}, \quad \frac{1}{q} < \frac{1}{4}, \quad \frac{1}{p_1} - \frac{1}{q} \geq \frac{2}{3},$$
$$\text{and } 0 \leq \frac{1}{p_2} - \frac{1}{q} \leq \frac{1}{3}, \quad (p_2, q) \notin \{(1,1), (1, \frac{3}{2}), (3, \infty), (\infty, \infty)\},$$

then, for any given $\omega \in \mathbb{R} \backslash \{0\}$ *there exists a distributional time-harmonic solution to the fully anisotropic Maxwell equations that satisfies*

$$\|(E, H)\|_{L^q(\mathbb{R}^3)} \lesssim_{p_1, p_2, q, \omega} \|(J_e, J_m)\|_{L^{p_1}(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)}.$$

If additionally $J_e, J_m \in L^q(\mathbb{R}^3)$, $q < \infty$, *then* $E, H \in W^{1,q}(\mathbb{R}^3)$ *is a weak solution satisfying*

References

- (w/ Roland Schnaubelt) Quasilinear Maxwell equations in two dimensions (arXiv:2105.06146, accepted to *Pure and Applied Analysis*)
- Well-posedness for Maxwell equations with Kerr nonlinearity in three dimensions via Strichartz estimates (arXiv:2108.07691)
- Resolvent estimates for time-harmonic Maxwell's equations in the partially anisotropic case (*Journal of Fourier Analysis and Applications*)
- (w/ Rainer Mandel) Time-harmonic solutions for Maxwell's equations in anisotropic media and Bochner-Riesz estimates with negative index for non-elliptic surfaces (*Ann. Henri Poincaré: JMTP*)

Outlook

- (w/ Roland Schnaubelt) Maxwell equations in the fully anisotropic case: Combine the phase space analysis with arguments for the fully anisotropic time-harmonic case
- Maxwell equations on domains?

Thank you for your attention!