Leading order asymptotics for fast diffusions on a bounded domain

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Joint work with

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The model.

We study solutions $w = w(\tau, x) \geq 0$ to the nonlinear diffusion equation or quasilinear heat equation

$$\partial_\tau w = \Delta w^m, \quad w(0) = w_0 \geq 0.$$

Depending on the value of $m > 0$, this is a model for

- diffusion of gas through a porous medium,
- population dynamics,
- gas kinetics,
- diffusion in plasma,
- certain geometric flows.

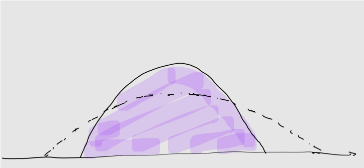
The archetype model for nonlinear diffusions.

Characteristic features: PME.

For $m > 1$, it features slow diffusion:

$$\partial_\tau w = \Delta w^m = \nabla \cdot (\underbrace{m\, w^{m-1}}_{\to 0 \text{ as } w \to 0} \nabla w).$$

Degenerate diffusion. Consequence: Finite speed of propagation, i.e., compactly supported solutions remain compactly supported.



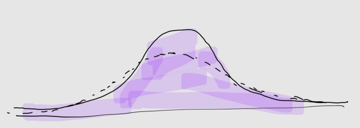
Well-posedness and regularity are well understood.

Characteristic features: FDE.

For $m < 1$, it features fast diffusion:

$$\partial_\tau w = \Delta w^m = \nabla \cdot (\underbrace{m\, w^{m-1}}_{\to\infty \text{ as } w\to 0} \nabla w).$$

Singular diffusion. Consequence: Infinite speed of propagation, i.e., solutions become positive instantaneously.



Well-posedness and regularity are well understood.

Focus in this talk:

Large-time behavior

Large-time dynamics in $\mathbb{R}^N$.

- Convergence to self-similar Barenblatt solution

$$W(\tau, x) = t^{-N\alpha} V(x/\tau^{\alpha}) :$$

  Kamin '73-'76, Friedman–Kamin '80, Kamin–Vázquez '88, Vázquez '03

- Rates of convergence: Carillo–Toscani '00, Otto '01, del Pino–Dolbeault '02, etc.
- Spectral analysis: Zel'dovitch–Barenblatt '58, Denzler–McCann '05, CS '13.
- Asymptotic expansions/Invariant manifolds: Angenent '88, Denzler–Koch–McCann '14, CS '15+, etc.

Slow diffusion in bounded domain Ω.

We impose Dirichlet boundary conditions: $w = 0$ on $\partial\Omega$.

- Convergence to separation-of-variables solution ("friendly giant")

$$W(\tau, x) = \tau^{-\beta} V(x),$$

  also sharp rates of convergence: Aronson–Peletier '81, Vázquez '04.

- In particular: $w(\tau) \to 0$ as $\tau \to \infty$.

Fast diffusion in bounded domain Ω.

We impose Dirichlet boundary conditions: $w = 0$ on $\partial\Omega$.

- Extinction in finite time of bounded solutions, $0 \leq w \leq C$,

$$\exists T = T(w_0) \quad \text{such that} \quad w(\tau) \to 0 \quad \text{as } \tau \uparrow T,$$

  cf. Sabinina '62, '65, Beryman–Holland '80.

- Convergence to a separation-of-variables solution

$$W(\tau, x) = ((1-m)(T-\tau))^{\frac{1}{1-m}} V(x)^{\frac{1}{m}},$$

  cf. Berryman–Holland '80, Feireisl–Simondon '00.

- Sharp rates of convergence (at least partial results): for $m \in (m_c, 1)$ with $m_c = \frac{N-2}{N+2}$ by Bonforte–Figalli '21; for $m = m_c$ and radial solutions on a ball: Galaktinov–King '02.

Rescaling.

Large time behavior best understood in rescaled variables.

Motivated by separation-of-variables solution,

$$W(\tau, x) = ((1-m)(T-\tau))^{\frac{1}{1-m}} V(x)^{\frac{1}{m}},$$

consider

$$w(\tau, x) = ((1-m)(T-\tau))^{\frac{1}{1-m}} v(t, x)^{\frac{1}{m}},$$

$$t = \log\left(\frac{T}{T-\tau}\right)^{\frac{m}{m-1}}, \quad p = \frac{1}{m} > 1,$$

which gives rise to

$$p^{-1}\partial_t v^p - \Delta v = v^p, \quad -\Delta V = V^p.$$

Goal: Find optimal rate of convergence for

$$v(t) \to V \quad \text{as } t \to \infty.$$

The stationary problem $-\Delta V = V^p$ in Ω.

- In general, solutions to elliptic PDE are not unique.
- However, large time limit uniquely determined by initial datum, $\exists! V = V(w_0)$, cf. Feireisl–Simondon '00.
- Formally, a stationary solution is a critical point of the Lyapunov functional

$$\frac{1}{2}\int_\Omega |\nabla V|^2\,dx - \frac{1}{p+1}\int_\Omega V^{p+1}\,dx.$$

- Sobolev sub-criticality $p < p_c = \frac{1}{m_c} = \frac{N+2}{N-2}$ implies compactness of embedding $W_0^{1,2}(\Omega) \subset L^{p+1}(\Omega)$.
- For $p \in (1, p_c)$, stationary solutions satisfy the boundary estimates

$$\mathrm{dist}(x, \partial\Omega) \lesssim V(x) \lesssim \mathrm{dist}(x, \partial\Omega).$$

- For $p > p_c$, existence of nonnegative bounded solutions may fail.

The relative error $h = \frac{v}{V} - 1$.

- Bonforte–Grillo–Vázquez '12: Uniform convergence of relative error:
$$\|h(t)\|_{L^\infty(\Omega)} = \|\frac{v(t) - V}{V}\|_{L^\infty(\Omega)} \to 0 \quad \text{as } t \to \infty.$$
- Change of perspective:
$$\partial_t h + Lh = N[h],$$
with
$$Lh = -V^{-1-p} \nabla \cdot (V^2 \nabla h) - (p-1)h,$$
$$N[h] = \underbrace{((1+h)^p - 1 - ph)}_{\lesssim |h|^2} + \underbrace{(1 - (1+h)^{p-1})\partial_t h}_{\lesssim |h||\partial_t h|}.$$
Recall: $V(x) \sim \mathrm{dist}(x, \partial\Omega)$, hence L becomes singular on $\partial\Omega$:
$$Lh = -\underbrace{V^{1-p}}_{\to\infty} \Delta h - 2 \underbrace{V^{-p}}_{\to\infty} \nabla V \cdot \nabla h - (p-1)h.$$
- No boundary conditions!

Understanding the difficulties: The spectrum.

Consider

$$Lh = -V^{-1-p}\nabla\cdot(V^2\nabla h) - (p-1)h.$$

- The operator L has a discrete spectrum on the Hilbert space L^2_{p+1} induced by

$$\|h\|^2_{L^2_\sigma} = \int_\Omega h^2\, d\mu_\sigma, \quad d\mu_\sigma = V^\sigma\, dx.$$

- For $h \equiv 1$, it holds that

$$Lh = L1 = 1 - p = (1-p)h.$$

  Hence, there are negative eigenvalues.

- It can be proved that $1 - p$ is the smallest eigenvalue. Equivalently, there is no Poincaré inequality of the type

$$\|h\|_{L^2_{p+1}} \leq C\|\nabla h\|_{L^2_2}.$$

Understanding the difficulties: The linear entropy approach.

Consider linear dynamics

$$\partial_t h + Lh = 0.$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h^2\, d\mu_{p+1} + \int_\Omega |\nabla h|^2\, d\mu_2 = (p-1)\int_\Omega h^2\, d\mu_{p+1}.$$

In the absence of a suitable Poincaré inequality, this identity implies not even decay! We would need

$$c\|h\|^2_{L^2_{p+1}} \leq \|\nabla h\|^2_{L^2_2} \quad \text{with } c > p-1.$$

Note:

$$(p-1)\|h\|^2_{L^2_{p+1}} = \|\nabla h\|^2_{L^2_2} \quad \text{iff} \quad h \in \ker L.$$

The Bonforte–Figalli result.

Main strategy:

- Nonlinear entropy method.
- Restrict to "generic" domains for which $\ker L = \{0\}$.
- Convergence in relative error, $\|h(t)\|_{L^\infty} \to 0$, ensures decay of unstable modes.
- Apply improved Poincaré inequality for stable modes.

Theorem (Bonforte–Figalli CPAM '21)

If Ω is generic, it holds that

$$\|h(t)\|_{L^2_{p+1}} \lesssim e^{-\lambda_s t},$$

where λ_s is the first positive eigenvalue of L.

Two questions:

What about “non-generic” domains?

AND

Do we know something about higher-order asymptotics?

"Non-generic" domains

Example (Coffman '84, Li '90, Byeon '97, Akagi–Kajikiya '14)

On *sufficiently small* annuli $\Omega = B_R \setminus B_r$, there exists symmetry breaking (i.e., non-radial) least energy solutions of

$$\frac{1}{2}\int_\Omega |\nabla V|^2\,dx - \frac{1}{p+1}\int_\Omega V^{p+1}\,dx.$$

Consequence: $\exists\{V_s\}_s$ stationary solutions with $\partial_s|_{s=0} V_s \neq 0$.

$$\implies \quad L(\partial_s|_{s=0} V_s/V_0) = 0.$$

Thus

$$K := \dim\ker L > 0.$$

A new approach for arbitrary smooth domains based on a

dynamical systems argument

Main result: A dichotomy

Theorem (Choi–McCann–CS '21+)

It EITHER holds that

$$\|h(t)\|_{L^\infty} \gtrsim \|h(t)\| \gtrsim \frac{1}{t} \quad as\ t \to \infty,$$

OR

$$\|h(t)\| \lesssim \|h(t)\|_{L^\infty} \lesssim e^{-\lambda_s t} \quad as\ t \to \infty.$$

2nd result: Integrable kernels

By Bonforte–Figalli '21: If $\ker L = \{0\}$, then the convergence is exponentially fast.

This can be extended:

Theorem (Choi–McCann–CS '21+)
If the kernel is integrable, then the convergence is exponentially fast.

Definition: A kernel is integrable if for every $\psi \in \ker L$, $\exists \{V_s\}_s$ stationary solutions with $\psi = \partial_s|_{s=0} V_s / V_0 \neq 0$.

The idea of the proof goes back to works by Allard–Almgren '81 and Simon '85 in the context of minimal surfaces and geometric evolution equations.

3rd result: Higher order asymptotics.

Theorem (Choi–McCann–CS '21+)

If the convergence is exponentially fast, then

$$h(t,x) = \sum_{\lambda_i \in [\lambda_s, 2\lambda_s)} C_i e^{-\lambda_i t} \varphi_i(x) + o\left(e^{-\max_i \lambda_i t}\right),$$

where $L\varphi_i = \lambda_i \varphi_i$ *and* $\{\varphi_i\}_i$ *is orthonormal,* $C_i \in \mathbb{R}$.

Idea of the proof of the dichotomy.

Key ingredient 1: New smoothing estimate

Proposition (Choi–McCann–CS '21+)

Suppose $\|h(t)\|_{L^\infty} \leq \varepsilon$ *for all* t. *Then, for any* $k \in \mathbb{N}_0$ *and* $t > 0$, *it holds*

$$\|\partial_t^k h(t)\|_{L^\infty} \leq C(k,t)\|h_0\|_{L^2_{p+1}}.$$

Main consequence: Quadratic estimate on nonlinearity:

$$\begin{aligned}\|N[h(t)]\|_{L^2_{p+1}} &\lesssim \left(\|h(t)\|_{L^\infty} + \|\partial_t h(t)\|_{L^\infty}\right)\|h(t)\|_{L^2_{p+1}} \\ &\lesssim \|h(t-1)\|_{L^2_{p+1}}\|h(t)\|_{L^2_{p+1}} \\ &\lesssim \varepsilon\|h(t)\|_{L^2_{p+1}}\end{aligned}$$

Spectral decomposition.

Orthogonal projection onto stable, center and unstable eigenspaces:

$$h_s = P_s h, \quad h_c = P_c h, \quad h_u = P_u h,$$
$$\partial_t Ph + LPh = PN[h].$$

Use smoothing estimates to get:

$$\frac{d}{dt}\|h_s\| + \underbrace{(\lambda_s - C\varepsilon)}_{\overset{!}{>}0}\|h_s\| \leq C\varepsilon\left(\|h_u\| + \|h_c\|\right),$$

$$\left|\frac{d}{dt}\|h_c\|\right| \leq C\varepsilon\left(\|h_s\| + \|h_u\| + \|h_c\|\right),$$

$$\frac{d}{dt}\|h_u\| + \underbrace{(\lambda_u + C\varepsilon)}_{\overset{!}{<}0}\|h_u\| \geq C\varepsilon\left(\|h_s\| + \|h_c\|\right),$$

where $\lambda_u < 0$ largest negative eigenvalue,

$\lambda_s > 0$ smallest positive eigenvalue.

Key ingredient 2. Known dynamical systems argument

Unstable modes cannot be active as

$$\|h(t)\| \to 0 \quad \text{as } t \to \infty$$

by Bonforte–Grillo–Vázquez '12.

Lemma (Merle–Zaag '98, dichotomy ODE lemma)

EITHER

$$\|h_u(t)\| + \|h_s(t)\| = o(\|h_c(t)\|) \quad \textit{as } t \to \infty,$$

OR

$$\|h_u(t)\| + \|h_c(t)\| = o(\|h_s(t)\|) \quad \textit{as } t \to \infty.$$

Remark: Original applied to classify connections between critical points for

$$\partial_t u - \Delta u = u^p.$$

If the stable modes dominate, $\|h(t)\| \lesssim \|h_s(t)\|$:

Starting point:

$$\frac{d}{dt}\|h_s\| + \underbrace{(\lambda_s - C\varepsilon)}_{\overset{!}{>}0}\|h_s\| \leq C\varepsilon\left(\|h_u\| + \|h_c\|\right) \leq \tilde{C}\varepsilon\|h_s\|,$$

gives exponential decay,

$$\|h(t)\| \lesssim \|h_s(t)\| \lesssim e^{-\tilde{\lambda}t},$$

if ε is sufficiently small, where $0 < \lambda_s - \tilde{\lambda} \lesssim \varepsilon$.

For optimal rate, iterate

$$\frac{d}{dt}\|h_s\| + \lambda_s\|h_s\| \lesssim \|N[h(t)]\| \lesssim \|h(t-1)\|\|h(t)\| \lesssim e^{-2\tilde{\lambda}t},$$

and use $2\tilde{\lambda} > \lambda_s$.

If the center modes dominate, $\|h(t)\| \lesssim \|h_c(t)\|$:

Starting point:

$$-\frac{d}{dt}\|h_c\| \lesssim \varepsilon\left(\|h_s\| + \|h_u\| + \|h_c\|\right) \lesssim \|h_c\|,$$

so that

$$\|h(t-1)\| \lesssim \|h_c(t-1)\| \lesssim \|h_c(t)\| \leq \|h(t)\|.$$

Therefore

$$-\frac{d}{dt}\|h_c\| \leq \|N[h(t)]\| \lesssim \|h(t-1)\|\|h(t)\| \lesssim \|h(t)\|^2 \lesssim \|h_c(t)\|^2,$$

which gives the algebraic lower bound

$$\|h(t)\| \geq \|h_c(t)\| \gtrsim 1/t.$$

Comment on the smoothing estimate

Proposition (Choi–McCann–CS '21+)

Suppose $\|h(t)\|_{L^\infty} \leq \varepsilon$ *for all* t. *Then, for any* $k \in \mathbb{N}_0$ *and* $t > 0$, *it holds*

$$\|\partial_t^k h(t)\|_{L^\infty} \leq C(k,t)\|h_0\|_{L^2_{p+1}}.$$

Main steps:

- L^2_{p+1}-based energy estimates,
- maximal regularity estimates, for $\partial_t h + Lh = f$ with $h(0) = 0$,

$$\|h\|_{L^2(L^2_{2p})} + \|\nabla h\|_{L^2(L^2)} + \|\nabla^2 h\|_{L^2(L^2_2)} \lesssim \|f\|_{L^2(L^2_{2p})},$$

- interpolation estimates,
- higher regularity in time and tangential directions,
- Sobolev embedding.

Thank you for your attention.