Quasi-invariance of Gaussian measures for the 2D Nonlinear Schrödinger equations

(joint with Yu Deng, Nikolay Tzvetkov)

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Nonlinear Schrödinger equations

Nonlinear Schrödinger equations (NLS):

$$i\partial_t u + \Delta u = \pm |u|^{p-1} u, \quad (t,x) \in \mathbb{R} \times \mathbb{T}^d, \ u(t,x) \in \mathbb{C}$$

Conserved Quantities:

$$M[u] = \int_{\mathbb{T}^d} |u|^2 dx, \quad E[u] = \int_{\mathbb{T}^d} \frac{1}{2} |\nabla u|^2 dx \pm \frac{1}{p+1} \int_{\mathbb{T}^d} |u|^{p+1} dx.$$

- Typical expected dynamical properties: Recurrence properties, Energy Cascade?
- Macroscopic description of the flow.
  - One way is to equip some "natural" probability measures and study their evolution along the NLS flow on $\mathbb{T}^d$, which is the main objective of this talk.
  - Sometimes macroscopic properties lead to dynamical consequences. For example, the existence of invariant measures implies the recurrence property of the flow, thanks to Poincaré.

Gaussian measures

- We will define a Gaussian measure μ_s, formally of the form

$$Z^{-1} e^{-\frac{1}{2}\|u\|_{H^s}^2} du = Z^{-1} \exp\Big(-\frac{1}{2} \sum_{k \in \mathbb{Z}^d} \langle k \rangle^{2s} |\widehat{u}_k|^2 \Big) \prod_{k \in \mathbb{Z}^d} d\widehat{u}_k.$$

- The above formal measure can be written as the limit of truncated Gaussian measures

$$\frac{1}{Z_N} \prod_{|k| \le N} e^{-\frac{1}{2}\langle k \rangle^{2s} |\widehat{u}_k|^2} d\widehat{u}_k.$$

- This indicates that μ_s can be induced by the randomization: Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\omega \mapsto \phi^\omega(x) := \sum_{k \in \mathbb{Z}^d} \frac{g_k(\omega)}{\langle k \rangle^s} e^{ik\cdot x},$$

  where $(g_k(\omega))_{k \in \mathbb{Z}^d}$ are i.i.d. complex Gaussian random variables on Ω, with mean 0 and variance 1.
- μ_s can be also identified with its covariance operator $\langle \nabla \rangle^{-2s}$ from $H^{-s} \mapsto H^s$.

Gaussian measures: sequel

Q: The measure μ_s is defined on which space?

- For $N < M$,

$$\mathbb{E}\Big[\Big\| \sum_{N\leq|k|\leq M} \frac{g_k(\omega)}{\langle k\rangle^s} e^{ik\cdot x}\Big\|^2_{H^\sigma(\mathbb{T}^d)}\Big] \sim \sum_{N\leq|k|\leq M} \frac{1}{\langle k\rangle^{2s-2\sigma}}$$

  converges if and only if

$$\sigma < s - \frac{d}{2}.$$

  We conclude that

$$\phi^\omega \in L^2(\Omega; H^\sigma(\mathbb{T}^d))$$

  for every $\sigma < s - \frac{d}{2}$. So μ_s is supported on

$$H^{(s-\frac{d}{2})-} := \bigcap_{\sigma < s-\frac{d}{2}} H^\sigma.$$

- Furthermore, $\mu_s(H^{s-\frac{d}{2}}(\mathbb{T}^d)) = 0$, in particular, $\mu_s(H^s) = 0$.
- There is a particular importance for the measure μ_1, related to the Gibbs measure.

The Gibbs measure Φ_d^{p+1} model:

- Defocusing Φ_d^{p+1} model corresponds to the Hamiltonian

$$\mathcal{H}[u] = \underbrace{\frac{1}{2}\int_{\mathbb{T}^d} |\nabla u|^2 dx}_{\mathcal{H}_{\mathrm{ki}[u]}} + \underbrace{\frac{1}{p+1}\int_{\mathbb{T}^d} |u|^{p+1} dx}_{V[u]}$$

  with formal expression $e^{-\mathcal{H}[u]}du$. The Gibbs measure is expected to be defined as $d\rho(u) = e^{-V[u]}d\mu_1(u)$, where $\mu_1 =$ "$e^{-\frac{1}{2}\|\nabla u\|_{L^2}^2}du$" is the Gaussian free field.
- The above construction is true only for $d = 1$, since for $d \geq 2$, the support of μ_1 $H^{(1-d/2)-}(\mathbb{T})$ misses $L^2(\mathbb{T}^d)$. For higher dimensions, we need renormalization for $V[u]$ to define the Gibbs measure. The renormalization changes the original Hamiltonian as well as its flow.

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- For the construction, seminar work by: Glimm-Jaffe, Lebowitz-Rose-Speer, Simon, Nelson, Wilson, Aizenman, Barashkov-Gubinelli,....
  - $d = 1, 2$, $\Phi_1^{p+1}, \Phi_2^{p+1}$ for any $p \in 2\mathbb{N} + 1$;
  - $d = 3$, Φ_3^4 (for other p ??);
  - $d \geq 4$, Φ_d^{p+1} cannot be done for any p (Aizenman, Duminil-Copin).
- Only for $d = 1, 2$, Φ_d^{p+1} is absolutely continuous with respect to μ_1.

Gaussian measures under transformations

Definition

Given a reversible flow $\varphi(t)$ and a Gaussian measure μ on some Banach space X, we say that μ is *quasi-invariant* along $\varphi(t)$ if $\varphi(t)_{\#}\mu \ll \mu$ for any $t \in \mathbb{R}$.

- At reasonable regularity level (e.x. 1D NLS), invariant Gibbs measure implies that the Gaussian free field μ_1 is *quasi-invariant* along the NLS flow: the transported measure $\Phi(t)_{\#}\mu_1 \ll \mu_1$ for $t \in \mathbb{R}$.
- In the infinite-dimensional space, transported measures become singular easily:
  - Cameron-Martin 1944: Let $f \in H^{\sigma}(\mathbb{T}^d)$ and μ_f be the image of the measure μ_s under the translation
    $$u \mapsto u + f$$
    on H^{σ}. Then $\mu_f \ll \mu_s$ if and only if $f \in H^s(\mathbb{T}^d)$ for $s > \sigma + \frac{d}{2}$. Correspondingly, the Radon-Nikodym density is
    $$e^{-\|f\|_{H^s}^2 - (u,f)_{H^s}}.$$
  - Oh-Sosoe-Tzvetkov: Consider the flowmap $\phi(t)$ defined by the ODE $i\partial_t u = |u|^2 u$. Then for any $t \neq 0$, $\phi(t)_{\#}\mu_s$ is singular to μ_s ($s \geq 1$).
- We now consider the specific flow defined by NLS. It turns out that the dispersion can prevents the measure to become singular.

Main result

Defocusing cubic NLS on $\mathbb{T}^2$:

$$i\partial_t u + \Delta u = |u|^2 u,\ (t,x) \in \mathbb{R} \times \mathbb{T}^2, \quad u|_{t=0} = u_0 \in H^\sigma.$$

- Scaling critical space $H^{s_c}(\mathbb{T}^2)$, $s_c = 0$. Locally well-posed in H^σ, $\sigma > 0$ (Bourgain).
- The flowmap $\Phi(t)$ is is globally defined on H^σ, for $\sigma \geq 1$, with the property that (Bourgain, Colliander-Kwon-Oh)

$$\|\Phi(t)u_0\|_{H^\sigma(\mathbb{T}^2)} \lesssim \langle t\rangle^{\alpha(\sigma)} C(\|u_0\|_{H^\sigma}).$$

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Theorem (Deng-S.-Tzvetkov, '21-'22)

For $s \geq 2$, the Gaussian measure μ_s is quasi-invariant along the cubic NLS flow $\Phi(t)$.

- supp$\mu_s = H^{(s-1)-}(\mathbb{T}^2)$ where $\Phi(t)$ is globally defined. The required regularity $s \geq 2$ is such that on supp(μ_s), the flow $\Phi(t)$ is globally well-defined.

Comparison for the 2D invariant Gibbs measure problem

Theorem (Bourgain '96, Deng-Nahmod-Yue '19)

There exists a full μ_1 measure (so full Φ_2^{2m+2} measure) set $\Sigma \subset H^{0-}(\mathbb{T}^2)$, such that the flow $:\Phi(t):$ of the renormalized NLS (Wick-ordered NLS):

$$i\partial_t u + \Delta u =: |u|^{2m}u := \text{``}\sum_k e^{ik\cdot x}\Big(\sum_{\substack{k_1 - k_2 + \cdots + k_{2m+1} = k \\ \text{no paring: } k_{2j-1} \neq k_{2l}}} \widehat{u}_{k_1}\overline{\widehat{u}}_{k_2}\cdots\widehat{u}_{k_{2m+1}}\Big)\text{''}$$

is well-defined on Σ. Moreover, the Gibbs measure Φ_2^{2m+2} is invariant along $:\Phi(t):$

- The Wick-ordering is necessary to define the nonlinearity on H^{0-} almost surely. Due to the low-regularity nature, the Cauchy problem is very difficult to solve!
- In an impressive work of DNY, they extend Bourgain's theorem from $m = 1$ to any m, by introducing the novel Random averaging operator theory to overcome an essential obstruction. Their method inspires many other works.
- However, all these invariant Gibbs measure theorems does not provide information on the transport properties for μ_1, under the real NLS flow.

Methodology I: Deterministic argument

Here we present several soft-analysis schemes developed in the works of Tzvetkov and: Gunaratnam, Oh, Planchon, Sosoe, Visciglia, Weber, ...

Formally, $d\mu_s(u) = \frac{1}{Z} e^{-\|u\|_{H^s}^2} du$, and we look for a suitable modified energy

$$E_s(u) := \|u\|_{H^s}^2 + R_s(u) \sim \|u\|_{H^s}^2$$

and look at the evolution of the measure (after suitable truncation)

$$d\rho_s(u) := e^{-R_s(u)} d\mu_s(u) \text{``} = \frac{1}{Z} e^{-E_s(u)} du''.$$

The Radon-Nikodym density is (if exists) then $e^{-(E_s(\Phi(t)u) - E_s(u))}$. Though $E_s(\Phi(t)u)$ and $E_s(u)$ are both strongly diverging on $\mathrm{supp}(\mu_s)$, the hope is to use some smoothing property (time oscillation) of the dispersive flow.

Denote by $G_s(\tau) = \frac{d}{dt} E_s(\Phi(t)u)|_{t=\tau}$:

- If we are able to show that

$$\Big| \int_0^t G_s(\tau) d\tau \Big| \le C(\mathcal{H}[u]) \|u\|_{H^{s-\frac{d}{2}-}}^{\theta}$$

  for some θ, then we are done (with the desired density if $\theta < 2$, otherwise we need a cutoff for $\|u\|_{H^{s-\frac{d}{2}-}}$).

Methodology II: Using the "random oscillation"

The second method is to exploit the random oscillation. Formally, if $\mu_s(A) = 0$ (hence $\rho_s(A) = 0$), we want to show that $\rho_s(\Phi(t)A) = 0$. We compute

$$\frac{d}{dt}\rho_s(\Phi(t)A)|_{t=t_0} = \frac{d}{dt}\int_{\Phi(t)(A)} d\rho_s(u)|_{t=t_0} = \int_A \frac{d}{dt}e^{-E_s(\Phi(t)u)}|_{t=t_0}du,$$

thanks to the Liouville theorem. Recalling that

$$G_s(t_0) = \frac{d}{dt}E_s(\Phi(t)u)|_{t=t_0},$$

the above identity equals to

$$\int_A G_s(t_0)e^{-E_s(\Phi(t_0)u)}du = \int_{\Phi(t_0)(A)} G_s(0)e^{-E_s(u)}du.$$

Then by Hölder, we have

$$|\frac{d}{dt}\rho_s(\Phi(t)A)|_{t=t_0}| \leq \|G_s(0)\|_{L^p(d\rho_s)}\rho_s(\Phi(t_0)(A))^{1-\frac{1}{p}},\ \forall p \geq 2.$$

Then if we are able to show that

$$\|G_s(0)\|_{L^p(\rho_s)} \leq Cp,\ \forall p \geq 2,$$

then by Yudovich-type argument, we deduce that $\rho_s(\Phi(t)A) = 0$ for any t.

Modified energy for NLS?

Write

$$v(t) = e^{-it\Delta}u(t), \quad v(t) = \sum_k v_k(t)e^{ik\cdot x}.$$

If $u(t)$ solves $i\partial_t u + \Delta u = |u|^2 u$, then

$$\partial_t v_k = \frac{1}{i} \sum_{k_1 - k_2 + k_3 = k} e^{-it\Phi(\vec{k})} v_{k_1} \overline{v}_{k_2} v_{k_3},$$

where

$$\Phi(\vec{k}) := |k_1|^2 - |k_2|^2 + |k_3|^2 - |k|^2 = 2(k_1 - k_2)\cdot(k_2 - k_3).$$

We have

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{H^s}^2 = -\frac{1}{4}\mathrm{Im} \sum_{\substack{k_1 - k_2 + k_3 - k_4 = 0 \\ k_2 \neq k_1, k_3}} \psi_{2s}(\vec{k}) e^{-it\Phi(\vec{k})} v_{k_1} \overline{v}_{k_2} v_{k_3} \overline{v}_{k_4},$$

$$\psi_{2s}(\vec{k}) = |k_1|^{2s} - |k_2|^{2s} + |k_3|^{2s} - |k_4|^{2s}.$$

Warming up: 1D analysis

Candidates for the Modified energy can be found by integration by part (Poincaré-Dulac normal form):

$$\sum_{\substack{k_1-k_2+k_3-k_4=0\\ k_2\neq k_1,k_3}} \psi_{2s}(\vec{k})e^{-it\Phi(\vec{k})}v_{k_1}\overline{v}_{k_2}v_{k_3}\overline{v}_{k_4} = \partial_t \underbrace{\Big(\sum_{\substack{k_1-k_2+k_3-k_4=0\\ k_2\neq k_1,k_3}} \frac{\psi_{2s}(\vec{k})}{-i\Phi(\vec{k})} e^{-it\Phi(\vec{k})}v_{k_1}\overline{v}_{k_2}v_{k_3}\overline{v}_{k_4} \Big)}_{\mathcal{N}_0(v)} - \sum_{\substack{k_1-k_2+k_3-k_4=0\\ k_2\neq k_1,k_3}} \frac{\psi_{2s}(\vec{k})}{-i\Phi(\vec{k})} e^{-it\Phi(\vec{k})}\partial_t(v_{k_1}\overline{v}_{k_2}v_{k_3}\overline{v}_{k_4})$$

When $d=1$, we have

$$\psi_{2s}(\vec{k}) = -\Big(\int_0^1\int_0^1 (\nabla^2|\cdot|^{2s})(k_4+\theta_1(k_2-k_3)-\theta_2(k_1-k_2))d\theta_1 d\theta_2\Big)(k_1-k_2)\cdot(k_2-k_3).$$

Thus $|\psi_{2s}(\vec{k})| \lesssim \max\{|k_1|,|k_2|,|k_3|\}^{2s-2}|\Phi(\vec{k})|$. Then we get, for $s\geq 2$,

$$\frac{d}{dt}\Big(\|v(t)\|_{H^s}^2 + \frac{1}{2}\mathrm{Im}\mathcal{N}_0(v)\Big) \lesssim 1 + \|v(t)\|_{H^{s-1}}^2\|v(t)\|_{H^{\frac{1}{2}+}}^4.$$

- Can be obtained to any nonlinearity $p\in 2\mathbb{N}+1$. There is a nice physical-space based proof by Planchon-Tzvetkov-Visciglia.

2D Analysis, the setup

$$\mathcal{N}_{0,t}(v) = \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \psi_{2s}(\vec{k})\frac{e^{-it\Phi(\vec{k})}}{-i\Phi(\vec{k})} v_{k_1}\overline{v}_{k_2} v_{k_3}\overline{v}_{k_4},$$

$$\mathcal{R}_{0,t}(v) = \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})=0}} \psi_{2s}(\vec{k}) e^{-it\Phi(\vec{k})} v_{k_1}\overline{v}_{k_2} v_{k_3}\overline{v}_{k_4}$$

$$\mathcal{R}_{1,1,t}(v) = 2\sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it\Phi(\vec{k})} \sum_{p_1-p_2+p_3=k_1} e^{-it\Phi(\vec{p})} v_{p_1}\overline{v}_{p_2} v_{p_3}\overline{v}_{k_2} v_{k_3}\overline{v}_{k_4},$$

$$\mathcal{R}_{1,2,t}(v) = -2\sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})\neq 0}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it\Phi(\vec{k})} \sum_{q_1-q_2+q_3=k_2} e^{it\Phi(\vec{q})} v_{k_1}\overline{v}_{q_1} v_{q_2}\overline{v}_{q_3} v_{k_3}\overline{v}_{k_4}.$$

Defining

$$E_{s,t}(v) := \frac{1}{2}\|v\|_{H^s}^2 + \frac{1}{4}\mathrm{Im}\mathcal{N}_{0,t}(v),$$

then along the NLS flow, we have

$$\frac{d}{dt}E_{s,t}(v) := \frac{1}{4}\mathrm{Im}\big[\mathcal{R}_{1,1,t}(v) + \mathcal{R}_{1,2,t}(v) - \mathcal{R}_{0,t}(v)\big]$$

Let us look at the simplest (resonant) term

$$\mathcal{R}_{0,t}(v) := \sum_{\substack{k_1-k_2+k_3-k_4=0\\ \Phi(\vec{k})=0}} \psi_{2s}(\vec{k}) e^{-it\Phi(\vec{k})} v_{k_1}\overline{v}_{k_2} v_{k_3}\overline{v}_{k_4}.$$

W.L.O.G., we assume that $v_{k_j} = \widehat{P_{N_j} v}(k_j)$ and $N_{(1)} \geq N_{(2)} \geq N_{(3)} \geq N_{(4)}$ are the rearrangement of N_1, N_2, N_3, N_4.

- $|\psi_{2s}(\vec{k})| \lesssim N_{(1)}^{2s-2} N_{(3)}^2$.
- We have

  $$|\mathcal{R}_{0,t}(v)| \lesssim N_{(1)}^{2s-2} N_{(3)}^2 \int_0^{2\pi} \int_{\mathbb{T}^2} e^{it\Delta} f_1 \cdot \overline{e^{it\Delta} f_2} e^{it\Delta} f_3 \cdot \overline{e^{it\Delta} f_4} dt dx,$$

  where $\widehat{f_j}(k_j) = |v_{k_j}|$.
- The space-time integral can be treated using the bilinear Strichartz estimate. Due to the unavoidable loss $N_{(3)}^{0+}$, we have

  $$|\mathcal{R}_{0,t}(v)| \lesssim \|\mathbf{P}_{N_{(1)}} v\|_{H^{s-1}} \|\mathbf{P}_{N_{(2)}} v\|_{H^{s-1}} \|\mathbf{P}_{N_{(3)}} v\|_{H^{2+}} \|\mathbf{P}_{N_{(4)}} v\|_{L^2}.$$

- No matter how large s is, the above estimate is not enough for our need, as $v \in H^{(s-1)-}$ almost surely. Nevertheless, we are ϵ-close to what we expect (for s large).

Exploiting the random oscillation

By Method II, what we are allowed reduce the estimate to $t = 0$ and average on the support of the measure. So we have access to the probability toolbox: Wiener chaos estimate: $l-$linear Gaussian sum:

$$\mathcal{T}_l := \sum_{k_1,\cdots,k_l} c_{k_1,\cdots,k_l} g_{k_1}^{\pm}(\omega)\cdots g_{k_l}^{\pm}(\omega),$$

for any $p \geq 2$,

$$\|\mathcal{T}_l\|_{L^p_\omega} \leq Cp^{\frac{l}{2}} \|\mathcal{T}_l\|_{L^2_\omega}.$$

- The pairing contributions $(k_1 = k_2, k_3 = k_4), (k_1 = k_4, k_2 = k_3)$ in $\mathcal{R}_{0,t}(v)$ disappear by taking the imaginary part, it is reduced to estimate

$$p^2 \Big\| \sum_{\substack{k_1-k_2+k_3-k_4=0,\\ k_2\neq k_1,k_3 \\ \Phi(\vec{k})=0}} \psi_{2s}(\vec{k}) \mathbf{1}_{|k_j|\sim N_j} \frac{g_{k_1}(\omega)\overline{g}_{k_2}(\omega) g_{k_3}(\omega)\overline{g}_{k_4}(\omega)}{\langle k_1\rangle^s \langle k_2\rangle^s \langle k_3\rangle^s \langle k_4\rangle^s} \Big\|_{L^2_\omega}$$

- Consider the worst case, say $N_1 \sim N_2 \gg N_3 + N_4 = O(1)$, the above quantity can be crudely bounded by $p^2 N_{(1)}^{2s-2} \cdot N_{(1)}^{-2s+1} = p^2 N_{(1)}^{-1}$.
- By interpolating with the deterministic bound in the last slide, we conclude that $\|\mathrm{Im}\mathcal{R}_{0,t}(v)|_{t=0}\|_{L^p_\omega} \leq Cp$.

- The treatment for $\mathcal{N}_{0,t}(v)$ follows from the similar analysis + resonance decomposition according to the value of $\Phi(\vec{k})$.
- However, the estimate for the second generations $\mathcal{R}_{1,j,t}(v), j = 1, 2$ requires another algebraic cancellation.
- The reason is that in the high-high-low-low-low-low regime, the most problematic contribution is the paring of two dominant frequencies living in different generations. These types of pairing prevent us to gain from the Winer chaos.

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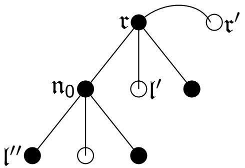
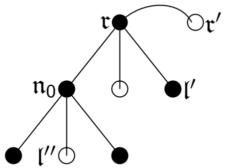
For example, in $\mathcal{R}_{1,1,t}(v)$, there are two types of pairings:



Paring the leaves $\mathfrak{l}', \mathfrak{l}''$



Paring the leaves $\mathfrak{l}', \mathfrak{l}''$

Key cancellation (Sequel)

Written in formula, these two pairing configurations are:

$$\mathcal{S}_{1,1,1}(v) := 4 \sum_{k_1 \neq k_2} |v_{k_2}|^2 \sum_{\substack{|k_3|+|k_4 \ll |k_1|,|k_2| \\ |p_2|+|p_3| \ll |k_1|,|k_2| \\ k_3 - k_4 = k_2 - k_1 \\ p_2 - p_3 = k_2 - k_1}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{-it(|k_3|^2 - |k_4|^2 + |p_2|^2 - |p_3|^2)} v_{k_3} \overline{v}_{k_4} \overline{v}_{p_2} v_{p_3}, \tag{1}$$

$$\mathcal{S}_{1,1,2}(v) := 4 \sum_{k_1, k_3} |v_{k_3}|^2 \sum_{\substack{|k_2|+|k_4| \ll |k_1|,|k_3| \\ |p_1|+|p_3| \ll |k_1|,|k_3| \\ p_1 + p_3 = k_1 + k_3 \\ k_2 + k_4 = k_1 + k_3}} \frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} e^{it(|k_2|^2 + |k_4|^2 - |p_1|^2 - |p_3|^2)} \overline{v}_{k_2} \overline{v}_{k_4} v_{p_1} v_{p_3}. \tag{2}$$

To understand the hidden cancellation, for $\mathcal{S}_{1,1,1}(v)$, one can think about the sum is taken over $|k_3|, |k_4|, |p_2|, |p_3| = O(1)$, then

$$\frac{\psi_{2s}(\vec{k})}{\Phi(\vec{k})} \approx \frac{|k_1|^{2s} - |k_2|^{2s}}{|k_1|^2 - |k_2|^2},$$

then the second sum in the definition of $\mathcal{S}_{1,1,1}$ is completely decoupled as $|\cdots|^2$ and we deduce that $\mathcal{S}_{1,1,1}$ is almost real.

Final remarks

- In work in progress with Y. Deng and N. Tzvetkov for the 3D NLS as well.
- For the moment, we do not know how much regularity we need to ensure the quasi-invariance property, especially in situations where we only have probabilistic well-posedness for the flow.
- What can we say about the Radon-Nikodym density? More philosophically, is there any link to the energy cascade phenomenon?

Thank you for your attention !