Symmetry Breaking for Ground States of Biharmonic Nonlinear Schrödinger Equations

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Joint work with

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$$\text{(NLS)} \qquad -\Delta u + \lambda u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \qquad p > 2, \quad \lambda > 0$$

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Solutions of (NLS) correspond, via the ansatz $\psi(t,x) = e^{i\lambda t}u(x)$, to standing wave solutions of the time-dependent semilinear Schrödinger equation

$$-i\frac{\partial \psi}{\partial t}(t,x) - \Delta\psi(t,x) = |\psi(t,x)|^{p-2}\psi(t,x),$$

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In the subcritical case $p < \left[\frac{2N}{N-2}\right]_+$, the stationary equation (NLS) has a variational structure w.r.t. the energy functional

$$E : H^1(\mathbb{R}^N) \to \mathbb{R}, \qquad E(u) := \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \lambda|u|^2\right)dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx.$$

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Classical result (Strauss 77, Gidas-Ni-Nirenberg 79,Kwong 89):
If $2 < p < \left[\frac{2N}{N-2}\right]_+$, then (NLS) admits a ground state solution which is unique, positive and radially symmetric up to a constant phase factor $e^{i\tau}$ and translations.

Important features of second order elliptic PDE

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I. Schwarz symmetrization $u \mapsto u^*$ preserves L^r-norms, $1 \leq r \leq \infty$ and satisfies

$$\int_{\mathbb{R}^N} |\nabla u^*|^2\,dx \leq \int_{\mathbb{R}^N} |\nabla u|^2\,dx \qquad \text{(Pólya-Szegö inequality)}$$

so it decreases the Sobolev quotient:

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Hence one may look for minimizers in the space $H^1_{rad}(\mathbb{R}^N)$.

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I. and II. are not available (in general) for higher order equations.

Biharmonic NLS

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The biharmonic nonlinear Schrödinger equation

$$\text{(BNLS)} \qquad \Delta^2 u + 2a\Delta u + bu = |u|^{p-2}u \quad \text{in } \mathbb{R}^N, \qquad a, b \in \mathbb{R},$$

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It has attracted growing attention recently partly due to the following reasons:

- To model stabilizing and self-focusing effects in the mass-supercritical regime for the NLS (Karpman & Shagalov 2000)
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$$u \mapsto E_{a,b}(u) = \underbrace{\frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta u|^2 - 2a|\nabla u|^2 + b|u|^2\right) dx}_{q_{a,b}(u)} - \frac{1}{p}\|u\|_p^p.$$

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Since

$$\mathsf{q}_{a,b}(u) = \int_{\mathbb{R}^N} \left[(|\xi|^2 - a)^2 + b - a^2\right] |\hat{u}(\xi)|^2 \, d\xi,$$

the quadratic form $\mathsf{q}_{a,b}$ is positive definite on $H^2(\mathbb{R}^N)$ iff

$$\text{(PD)} \qquad b > a^2 \quad \text{or} \quad 0 < b \le a^2 \quad \text{and} \quad a \le 0.$$

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In this range of parameters, we have, by Sobolev embeddings,

$$R_{a,b}(p) := \inf_{u \in H^2(\mathbb{R}^N) \setminus \{0\}} \frac{\mathsf{q}_{a,b}(u)}{\|u\|_p^2} > 0 \qquad \text{if } 2 < p < 2^*.$$

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A classical analysis of compactness up to translations (based on Lions' Lemma and the Brezis-Lieb Lemma) leads to the following

Theorem: Suppose $N \ge 1$, $2 < p < 2^*$, and that (PD) holds.
Then $R_{a,b}(p)$ is attained by a (ground state) solution of (BNLS).
Moreover, every ground state solution is real-valued up to a phase factor $e^{i\tau}$.

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Shape and symmetry of ground state solutions?

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We first consider the cooperativity condition

$$(CC) \qquad a < 0 \qquad \text{and} \qquad 0 < b < a^2$$

Theorem (Bonheure,Casteras, Moreira dos Santos & Nascimento 2018)

If (CC) holds, then every ground state solution of (BNLS) is positive up to a phase factor $e^{i\tau}$ and radially symmetric up to translations.

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Key property: As a consequence of (CC), the equation (BNLS) can be written as a cooperative system

$$(CS) \qquad (-\Delta + \lambda_1)u = v, \qquad (-\Delta + \lambda_2)v = |u|^{p-2}u \quad \text{in } \mathbb{R}^N.$$

with $\lambda_1, \lambda_2 > 0$ satisfying $\lambda_1 + \lambda_2 = -2a$ and $\lambda_1\lambda_2 = b$.
(Busca & Sirakov 2000)

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Condition (NC) also implies that radial solutions of (BNLS) are sign-changing (Bonheure,Casteras, Moreira dos Santos, Nascimento, 2018).

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Applying general Fourier symmetrization results of Benguria, Lenzmann & Sok, we observe the following:
If $p \in 2\mathbb{N}$, then, up to translations, ground state solutions u of (BNLS) are even functions, i.e., $u(-x) = u(x)$ for $x \in \mathbb{R}^N$.

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This is 'radial symmetry' if $N = 1$.

Main question for the present talk: What about the case $N \geq 2$?

Main result on symmetry breaking

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We consider the mixed dispersion case $a > 0$, $b > a^2$ in the following.

By rescaling we may assume $a = 1$.

With $\varepsilon := b - 1$, we may then rewrite (BNLS) as

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Theorem (Lenzmann, W. 21)

Let $N \geq 2$ and $2 < p < 2_* := \frac{2(N+1)}{N-1}$.

Then there exists $\varepsilon_0 = \varepsilon_0(p) > 0$ with the property that every ground state solution $u \in H^2(\mathbb{R}^N) \setminus \{0\}$ of (B_ε) is a nonradial function if $0 < \varepsilon \leq \varepsilon_0$.

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Remarks

- Since $(B)_\varepsilon$ is invariant under rotations and translations, there is no direct indication of the presence and form of symmetry breaking.
- In the case $N = 2$, $p = 4$, ground states are nonradial but even up to translations.
- Open question: Are ground states axially symmetric (up to translation)?

More symmetry breaking: Energy minimizers with fixed mass

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$$(B)_\varepsilon \qquad \Delta^2 u + 2\Delta u + (1+\varepsilon)u = |u|^{p-2}u, \qquad u \in H^2(\mathbb{R}^N).$$

Solutions of $(B)_\varepsilon$ also arises as the Euler-Lagrange equation associated with the minimization problem for the energy functional

$$\tilde{E} : H \to \mathbb{R}, \qquad \tilde{E}(u) = \int_{\mathbb{R}^N} |\Delta u|^2 \, dx - 2\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{2}{p}\int_{\mathbb{R}^N} |u|^p \, dx$$

restricted to the L^2-sphere

$$S(m) := \left\{ u \in H \ : \ \int_{\mathbb{R}^N} |u|^2 \, dx = m \right\}. \qquad \text{('fixed mass constraint')}$$

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From the dynamical point of view, this minimization problem is more natural then the one for the Sobolev quotient.

Both $\tilde{E}$ and $S(m)$ are invariant under the corresponding biharmonic nonlinear Schrödinger flow.

$\Longrightarrow$ Orbital stability properties of the set of minimizers of $\tilde{E}|_{S(m)}$.

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$S(m) := \Big\{ u \in H \,:\, \int_{\mathbb{R}^N} |u|^2\,dx = m \Big\}.$ ('fixed mass constraint')

The following result provides a link to ground state solutions of $(B)_\varepsilon$.

Theorem (Fernández, Jeanjean, Mandel & Mariș, 2021)

For every $m > 0$, the infimum of $\tilde{E}$ on $S(m)$ is attained in the mass-subcritical case

$$2 < p < \max(4, \frac{2(N+5)}{N+1}), \qquad p < 2 + \frac{8}{N}.$$

Moreover, every minimizer $u \in S(m)$ is a ground state solution of $(B)_\varepsilon$ for some $\varepsilon = \varepsilon(m)$, whereas $\varepsilon(m) \to 0^+$ as $m \to 0$.

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Corollary (Lenzmann & W.)

Let $N \geq 2$, and suppose that

$$2 < p < \frac{14}{3} \quad \text{if } N = 2 \qquad \text{and} \qquad 2 < p < 2_* \quad \text{if } N \geq 3.$$

Then there exists $m_0 = m_0(p) > 0$ with the property that for every $0 < m < m_0(p)$ all minimizers of $\tilde{E}$ on $S(m)$ are nonradial functions.

Symmetry breaking for the Dirichlet problem in the unit ball

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Consider the associated Dirichlet problem in the unit ball $B = B_1(0)$:

$$(\mathrm{DP}) \qquad \begin{cases} \Delta^2 u + 2a\Delta u + bu = |u|^{p-2}u & \text{in } B, \\ u = \partial_\nu u = 0 & \text{on } \partial B. \end{cases}$$

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Let $N \geq 2$ and $2 < p < 2_*$. Then, for $\varepsilon > 0$ sufficiently small, there exists $a_0 = a_0(\varepsilon, p) > 0$ with the property that every ground state solution $u \in H$ of (DP) is a nonradial function if $a > a_0$ and $b = (1+\varepsilon)a^2$.

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Symmetry/symmetry breaking is largely open in the remaining cases.

Symmetry breaking for (B_ε): Idea of the proof

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We need asymptotic expansions of $R_\varepsilon(p)$ and $R_\varepsilon^{rad}(p)$ in the limit $\varepsilon \to 0^+$.

'Nonradial' expansions

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$$\tau_{N,p} := \begin{cases} \frac{3}{4} + \frac{1}{2p} - \frac{N}{2}(\frac{1}{2} - \frac{1}{p}) & \text{if } 2 < p < 2_*; \\ \frac{1}{2} & \text{if } 2_* < p < 2^*. \end{cases}$$

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We have $\quad R_\varepsilon(p) = O(\varepsilon^{\tau_{N,p}}) \quad$ as $\varepsilon \to 0^+$.

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Moreover, if $2_* \leq p < 2^*$, we have

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'Nonradial' expansions: Special case $N = 2$

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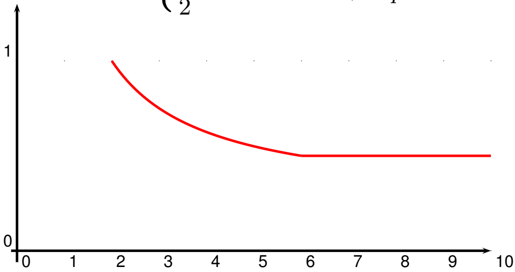
In the case $N = 2$ we have

$$2_* = 6, \qquad 2^* = \infty, \qquad \tau_{N,p} := \begin{cases} \frac{1}{4} + \frac{3}{2p}, & \text{if } 2 < p < 2_*; \\ \frac{1}{2} & \text{if } 2_* < p < 2^*. \end{cases}$$



The borderline exponent $p = 2_*$ hints at the **Stein-Tomas inequality.**

The Stein-Tomas inequality

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Let $N \geq 2$, $\mathcal{S} := S^{N-1}$ be the unit sphere in $\mathbb{R}^N$, and let $p \in [2_*, \infty)$. Then

$$\mathsf{C}_{ST}(p) := \inf_{w \in L^2(\mathcal{S}) \setminus \{0\}} \frac{\|w\|^2_{L^2(\mathcal{S})}}{\|\check{w}\|_p^2} > 0,$$

where, for $w \in L^2(\mathcal{S})$, the function $\check{w} \in L^p(\mathbb{R}^N)$ is a. e. given by

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1. The exponent bound $p \geq 2_*$ is sharp for this inequality (Knapp's example)
2. The constants $\mathsf{C}(p)$ in the expansion for $R_\varepsilon(p)$ are related to $\mathsf{C}_{ST}(p)$.

In particular: $\mathsf{C}(p) = \frac{2}{\pi} \mathsf{C}_{ST}(p) \quad$ if $2_* \leq p < 2^*$.

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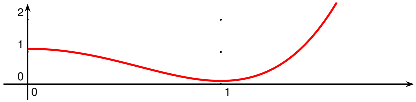

As $\varepsilon \to 0^+$, Fourier transforms of minimizers are expected to concentrate near the unit sphere $\mathcal{S} = \{|\xi| = 1\}$

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- Real-valued minimizers are constant if $3 \leq N \leq 7$ and $p \in 2\mathbb{N}$, $p \geq 4$ (Foschi 2015, Carneiro & Oliveira e Silva 2015, Oliveira e Silva & Quilodran 2021)

Expansion of minimal Sobolev quotients: The case $2_* \le p < 2^*$.

Claim: $$R_\varepsilon(p) = \frac{2}{\pi}\mathsf{C}_{ST}(p)\sqrt{\varepsilon} + o\left(\sqrt{\varepsilon}\right) \qquad \text{if } 2_* \le p < 2^*.$$

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Choose an almost extremal function $w \in L^2(\mathcal{S})$ in the Stein-Thomas inequality and test the Sobolev quotient with

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Apply ST-inequality to the functions $\widehat{u}_\varepsilon(r(\cdot)) \in L^2(\mathcal{S})$, $1-\delta \le r \le 1+\delta$.

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and hence

$$\begin{aligned} R_\varepsilon(p) \ge R_\varepsilon(2_*)^{\alpha}\varepsilon^{1-\alpha} &= \Big(\frac{2}{\pi}\mathsf{C}_{ST}(p)\sqrt{\varepsilon}\Big)^{\alpha}\varepsilon^{1-\alpha} + \text{h.o.t.} \\ &= \Big(\frac{2}{\pi}\mathsf{C}_{ST}(p)\Big)^{(N+1)\left(\frac{1}{2}-\frac{1}{p}\right)}\varepsilon^{\frac{3}{4}+\frac{1}{2p}-\frac{N}{2}(\frac{1}{2}-\frac{1}{p})} + \text{h.o.t.}. \end{aligned}$$

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Surprisingly, this is already the optimal exponent $\tau(N,p) = \frac{3}{4} + \frac{1}{2p} - \frac{N}{2}(\frac{1}{2} - \frac{1}{p})$.

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If $p < 2_* = \frac{2(N+1)}{N-1}$, these functions satisfy

$$\frac{\|w_\delta\|^2_{L^2(\mathcal{S})}}{\|\check{w}_\delta\|^2_p} = O(\delta^{\frac{N+1}{p} - \frac{N-1}{2}}) \to 0 \quad \text{as } \delta \to 0.$$

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Moreover, a pointwise estimate shows that, for suitable $c_1, c_2 > 0$,

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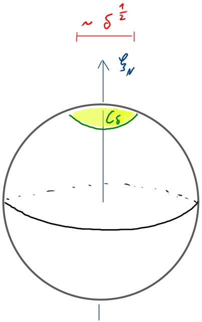

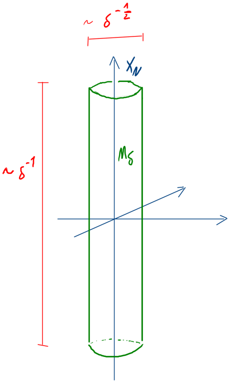

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Inspired by Knapp's example, we estimate $R_\varepsilon(p)$ with the ε-dependent test functions

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Using the fact that the Fourier symbol g_ε has a nondegenerate minimum at 1, we estimate

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Next step: Lower estimates for $R_\varepsilon^{rad}(p)$

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To obtain a sufficient asymptotic lower bound for

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For radial functions, we may repeat previous estimates with $\mathsf{C}_{ST}(p)$ replaced by $\mathsf{C}_{ST}^{rad}(p)$.

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Proposition

(i) If $2_*^{rad} < p \leq 2^*$, we have

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$$R_\varepsilon^{rad}(p) \ge C\varepsilon^\beta + o(\varepsilon^\beta) \qquad \text{as } \varepsilon \to 0^+$$

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$$\beta \in \begin{cases} \left(1 - N(\frac{1}{2} - \frac{1}{p}), \frac{1}{2} + \frac{1}{p}\right) & \text{in the case } N \le 4, \\ \left(1 - N(\frac{1}{2} - \frac{1}{p}), 1 - \frac{N}{4}(\frac{1}{2} - \frac{1}{p})\right) & \text{in the case } N \ge 5 \end{cases}$$

with a constant $C = C(N, p, \beta) > 0$.

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Here, again, (ii) is obtained from (i) by interpolating with the inequality

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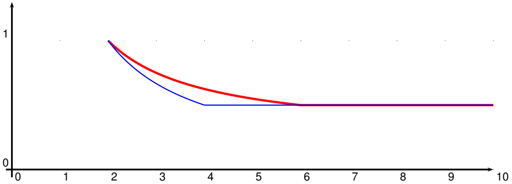
Hence every ground state solution of (B_ε) is nonradial in this case.

Special case $N = 2$: Recall that $2_*^{rad} = 4$ and $2_* = 6$.



Many thanks!